<u>Trigonometric Identities and Equations</u>

I. Fundamental Trigonometric Identities

A. Reciprocal identities

1. $\sec\theta = \dfrac{1}{\cos\theta}$ 2. $\csc\theta = \dfrac{1}{\sin\theta}$ 3. $\cot\theta = \dfrac{1}{\tan\theta}$

B.Quotient identities

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ 2. . $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

C. Pythagorean identities

1. $\sin^2\theta + \cos^2\theta = 1$ 2. $\tan^2\theta + 1 = \sec^2\theta$ 3. $1 + \cot^2\theta = \csc^2\theta$

D. Sum and difference identities

1. $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

2. $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

3. $\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$

E. Double angle identities

1. $\sin 2\theta = 2\sin\theta\cos\theta$

2. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

3. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

F. Half angle identities

1. $\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1 - \cos\theta}{2}$ 2. $\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1 + \cos\theta}{2}$ 3. $\tan^2\left(\dfrac{\theta}{2}\right) = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

G. Miscellaneous identities

1. $\sin(-\theta) = -\sin\theta$

2. $\cos(-\theta) = \cos\theta$

3. $\tan(-\theta) = -\tan\theta$

4. $\sin\theta \pm \sin\varphi = 2\sin\left(\frac{\theta \pm \varphi}{2}\right)\cos\left(\frac{\theta \mp \varphi}{2}\right)$

5. $\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta + \varphi}{2}\right)\cos\left(\frac{\theta - \varphi}{2}\right)$

6. $\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta + \varphi}{2}\right)\sin\left(\frac{\theta - \varphi}{2}\right)$

7. $\sin\theta\cos\varphi = \frac{1}{2}\sin(\theta + \varphi) + \frac{1}{2}\sin(\theta - \varphi)$

8. $\cos\theta\sin\varphi = \frac{1}{2}\sin(\theta + \varphi) - \frac{1}{2}\sin(\theta - \varphi)$

9. $\cos\theta\cos\varphi = \frac{1}{2}\cos(\theta + \varphi) + \frac{1}{2}\cos(\theta - \varphi)$

10. $\sin\theta\sin\varphi = \frac{1}{2}\cos(\theta - \varphi) - \frac{1}{2}\cos(\theta + \varphi)$

11. $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$

H. Useful suggestions for proving trigonometric identities

1. Avoid aimless transformations. Any transformation that is made in one of the members should lead in some way to the form of the other.
2. Start with the more complicated member of the identity and transform it into the form of the simpler member.
3. Where possible, express different functions in terms of the same function.
4. It is often useful to express all functions in terms of sines and cosines, or in terms of tangents and secants.

5. As a rule, trigonometric functions of a double angle, a half angle, or the sums and differences of angles should be expressed in terms of functions of the single angle.
6. Simplify expressions by utilizing basic identities and combining like terms.
7. Simplify fractions. For example, transform complex fractions into simple fractions or divide the terms of a fraction by the common factors.

I. Examples

1. Using trigonometric identities and fundamental trigonometric function values, find each of the following:

(a) $\sin 15^\circ = \sin\left(\frac{30^\circ}{2}\right) = \sqrt{\frac{1-\cos 60^\circ}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$

(b) $\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) -$

$$\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(c) $2\sin 15^\circ \cos 15^\circ = \sin[2(15^\circ)] = \sin 30^\circ = \frac{1}{2}$

(d) $\tan 15^\circ = \tan\left(\frac{30^\circ}{2}\right) = \frac{\sin 30^\circ}{1+\cos 30^\circ} = \frac{1/2}{1+\sqrt{3}/2} = \frac{1}{2+\sqrt{3}} = 2-\sqrt{3}$

(e) $\cos 37.5^\circ \cos 7.5^\circ = \frac{1}{2}\cos(37.5^\circ + 7.5^\circ) + \frac{1}{2}\cos(37.5^\circ - 7.5^\circ) =$

$$\frac{1}{2}\cos 45^\circ + \frac{1}{2}\cos 30^\circ = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}+\sqrt{3}}{4}$$

2. Prove: $\dfrac{1+\sec x}{\csc x} = \sin x + \tan x$

$$\frac{1+\sec x}{\csc x} = \frac{1}{\csc x} + \frac{\sec x}{\csc x} = \sin x + \frac{1/\cos x}{1/\sin x} = \sin x + \left(1/\cos x\right)\left(\sin x/1\right) = \sin x +$$

$$\frac{\sin x}{\cos x} = \sin x + \tan x$$

3. Prove: $\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x} = \dfrac{2}{\sin 2x}$

$$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{\cos x(\cos x)}{\sin x \cos x} + \frac{\sin x(\sin x)}{\sin x \cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} =$$

$$\frac{2}{2\sin x \cos x} = \frac{2}{\sin 2x}$$

4. Prove: $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

$$\frac{\tan x + \tan y}{\tan x - \tan y} = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}} = \frac{\left(\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}\right)(\cos x)(\cos y)}{\left(\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}\right)(\cos x)(\cos y)} =$$

$$\frac{\sin x \cos y + \sin y \cos x}{\sin x \cos y - \sin y \cos x} = \frac{\sin(x+y)}{\sin(x-y)}$$

II. Solution of Trigonometric Equations

A. Useful suggestions for solving trigonometric equations

1. Simplify the equation by clearing fractions, removing parentheses, combining like terms, and removing radicals.
2. Express functions of a double angle, a half angle, or the sums and differences of angles in terms of functions of the single angle; then express the different functions of the single angle in terms of a single function of that angle.

3. Solve the resulting equation, whether it be linear or quadratic in nature, for all the values of the angle in the given domain.
4. Checks the results by substituting into the original equation.

B. Examples

1. Solve for x in the interval $[0,2\pi)$: $2\sin x+\sqrt{3}=0$

   $2\sin x+\sqrt{3}=0 \Rightarrow 2\sin x=-\sqrt{3} \Rightarrow \sin x=\dfrac{-\sqrt{3}}{2}$. Get the reference angle $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{\pi}{3}(60^\circ)$. Since $\sin x$ is negative, x lies in the 3rd and 4th quadrants. Thus, $x=\pi(180^\circ)+\dfrac{\pi}{3}(60^\circ)=\dfrac{4\pi}{3}(240^\circ)$ or $x=2\pi(360^\circ)-\dfrac{\pi}{3}(60^\circ)=\dfrac{5\pi}{3}(300^\circ)$. Both of these values do check.

2. Solve for x in the interval $[0,2\pi)$: $\cos x=\cos x\tan x$

   $\cos x=\cos x\tan x \Rightarrow \cos x-\cos x\tan x=0 \Rightarrow \cos x(1-\tan x)=0 \Rightarrow$ $\cos x=0$ or $1-\tan x=0 \Rightarrow \cos x=0$ or $\tan x=1$. $\cos x=0 \Rightarrow x=\dfrac{\pi}{2}(90^\circ)$ or $x=\dfrac{3\pi}{2}(270^\circ)$. $\tan x=1 \Rightarrow$ reference angle = $\tan^{-1}(1)=\dfrac{\pi}{4}(45^\circ)$. Since $\tan x$ is positive, x lies in the 1st and 3rd quadrants. Thus, $x=\dfrac{\pi}{4}(45^\circ)$ or $x=\pi(180^\circ)+\dfrac{\pi}{4}(45^\circ)=\dfrac{5\pi}{4}(225^\circ)$. Thus, the solutions are $x=\dfrac{\pi}{4},\dfrac{\pi}{2},\dfrac{5\pi}{4}, or\ \dfrac{3\pi}{2}$

   and they all check.

3. Solve for x in the interval $[0,2\pi)$: $2\cos 3x=1$

   $2\cos 3x=1 \Rightarrow \cos 3x=\dfrac{1}{2}$. Since $0\le x<2\pi$, $0\le x<6\pi$. The reference angle for $3x$ is $\cos^{-1}\left(\dfrac{1}{2}\right)=\dfrac{\pi}{3}(60^\circ)$ and $\cos x$ is positive in the 1st and 4th quadrants. Thus, $3x=\dfrac{\pi}{3}(60^\circ)$, $3x=2\pi(360^\circ)-\dfrac{\pi}{3}(60^\circ)=\dfrac{5\pi}{3}(300^\circ)$, $3x=2\pi(360^\circ)+$

$\frac{\pi}{3}(60^\circ) = \frac{7\pi}{3}(420^\circ)$, $3x = 4\pi(720^\circ) - \frac{\pi}{3}(60^\circ) = \frac{11\pi}{3}(660^\circ)$, $3x = 4\pi(720^\circ) +$

$\frac{\pi}{3}(60^\circ) = \frac{13\pi}{3}(780^\circ)$, and $3x = 6\pi(1080^\circ) - \frac{\pi}{3}(60^\circ) = \frac{17\pi}{3}(1020^\circ) \Rightarrow$

$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, or\ \frac{17\pi}{9}$ and they all check.

4. Solve for x in the interval $[0, 2\pi)$: $\cos 2x = \cos x$

$\cos 2x = \cos x \Rightarrow \cos 2x - \cos x = 0 \Rightarrow (2\cos^2 x - 1) - \cos x = 0 \Rightarrow 2\cos^2 x -$

$\cos x - 1 = 0 \Rightarrow (2\cos x + 1)(\cos x - 1) = 0 \Rightarrow 2\cos x + 1 = 0$ or $\cos x - 1 = 0 \Rightarrow$

$\cos x = -\frac{1}{2}$ or $\cos x = 1$. $\cos x = -\frac{1}{2} \Rightarrow$ reference angle is $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and

x lies in the 2nd or 3rd quadrants since $\cos x$ is negative $\Rightarrow x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or

$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$. $\cos x = 1 \Rightarrow x = 0$. Thus, $x = 0, \frac{2\pi}{3}, or\ \frac{4\pi}{3}$ and they all check.

5. Solve for x in the interval $[0, 2\pi)$: $\sin x = \cos x$

$\sin x = \cos x \Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1 \Rightarrow$ reference angle is $\tan^{-1}(1) = \frac{\pi}{4}$ and

x lies in the 1st or 3rd quadrants since $\tan x$ is positive $\Rightarrow x = \frac{\pi}{4}$ or $x = \pi +$

$\frac{\pi}{4} = \frac{5\pi}{4}$ and they both check.

Practice Sheet – Trigonometric Identities and Equations

I. Verify the following identities:

(1) $\dfrac{1+\sec x}{\csc x}=\sin x+\tan x$

(2) $\tan\left(\dfrac{\pi}{4}+x\right)=\dfrac{\cos 2x}{1-\sin 2x}$

(3) $\csc 2x=\dfrac{\sec x}{2\sin x}$

(4) $\sin^2\left(\dfrac{x}{2}\right)=\dfrac{\sec x-1}{2\sec x}$

(5) $\dfrac{\cos 4x-\cos 2x}{\sin 2x-\sin 4x}=\tan 3x$

II. Solve the following equations for all values of x in the interval $[0, 2\pi\ )$:

(1) $3\sin x-4=5\sin x-3$

(2) $2\sin x\cos x=\sqrt{3}\cos x$

(3) $4\cos^2 x-1=0$

(4) $2\cos^2 x+3\sin x=3$

(5) $\sin x-\cos x=1$

Solution Key for Trigonometric Identities and Equations

I. (1) $\dfrac{1+\sec x}{\csc x}=\dfrac{1}{\csc x}+\dfrac{\sec x}{\csc x}=\sin x+\dfrac{1/\cos x}{1/\sin x}=\sin x+\left(\dfrac{1}{\cos x}\right)\left(\dfrac{\sin x}{1}\right)=\sin x+\tan x$

(2) $\tan\left(\dfrac{\pi}{4}+x\right)=\dfrac{\tan\left(\pi/4\right)+\tan x}{1-\tan\left(\pi/4\right)\tan x}=\dfrac{1+\tan x}{1-\tan x}=\dfrac{1+\dfrac{\sin x}{\cos x}}{1-\dfrac{\sin x}{\cos x}}=\dfrac{\cos x+\sin x}{\cos x-\sin x}=$

$$\frac{(\cos x+\sin x)(\cos x-\sin x)}{(\cos x-\sin x)(\cos x-\sin x)}=\frac{\cos^2 x-\sin^2 x}{\cos^2 x-2\cos x\sin x+\sin^2 x}=\frac{\cos 2x}{1-\sin 2x}$$

(3) $\csc 2x = \frac{1}{\sin 2x} = \frac{1}{2\sin x \cos x} = \frac{1}{2\sin x} \cdot \frac{1}{\cos x} = \frac{1}{2\sin x} \cdot \sec x = \frac{\sec x}{2\sin x}$

(4) $\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{2} = \frac{\frac{1-\cos x}{\cos x}}{\frac{2}{\cos x}} = \frac{\frac{1}{\cos x} - \frac{\cos x}{\cos x}}{2\sec x} = \frac{\sec x - 1}{2\sec x}$

(5) $\frac{\cos 4x - \cos 2x}{\sin 2x - \sin 4x} = \frac{-2\sin\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)}{2\sin\left(\frac{2x-4x}{2}\right)\cos\left(\frac{2x+4x}{2}\right)} = \frac{-2\sin 3x \sin x}{2\sin(-x)\cos 3x} =$

$\frac{-2\sin 3x \sin x}{-2\sin x \cos 3x} = \frac{\sin 3x}{\cos 3x} = \tan 3x$

II. (1) $3\sin x - 4 = 5\sin x - 3 \Rightarrow \sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$ and they both check.

(2) $2\sin x \cos x = \sqrt{3}\cos x \Rightarrow \sin x = \frac{\sqrt{3}}{2}$ or $\cos x = 0 \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$ or $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and they all check.

(3) $4\cos^2 x - 1 = 0 \Rightarrow \cos x = \pm\frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ and they all check.

(4) $2\cos^2 x + 3\sin x = 3 \Rightarrow 2(1-\sin^2 x) + 3\sin x = 3 \Rightarrow 2\sin^2 x - 3\sin x + 1 = 0 \Rightarrow$ $(2\sin x - 1)(\sin x - 1) = 0 \Rightarrow \sin x = \frac{1}{2}$ or $\sin x = 1 \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $x = \frac{\pi}{2}$ and they all check.

(5) $\sin x - \cos x = 1 \Rightarrow \sin x = 1 + \cos x \Rightarrow (\sin x)^2 = (1+\cos x)^2 \Rightarrow \sin^2 x = 1 +$ $2\cos x + \cos^2 x \Rightarrow 1 - \cos^2 x = 1 + 2\cos x + \cos^2 x \Rightarrow 2\cos^2 x + 2\cos x = 0 \Rightarrow$ $2\cos x(\cos x + 1) = 0 \Rightarrow \cos x = 0$ or $\cos x = -1 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = \pi$.

$x = \frac{\pi}{2}$ or $x = \pi$ are the solutions because they both check. However, $x = \frac{3\pi}{2}$ does not check in the original equation and thus is not a solution. [Note: $x = \frac{3\pi}{2}$ is an extraneous root created by squaring both sides of the original equation.]