



Existence and multiplicity of positive solutions for classes of singular elliptic PDEs [☆]

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ABSTRACT

We consider the boundary value problem

$$-\Delta u = \phi g(u)u^{-\alpha} \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ϕ is a nonnegative function in $L^\infty(\Omega)$ such that $\phi > 0$ on some subset of Ω of positive measure, $g : [0, \infty) \rightarrow \mathbb{R}$ is continuous. We establish the existence of three positive solutions when $g(0) > 0$ (positone), the graph of $\frac{s^{\alpha+1}}{g(s)}$ is roughly S-shaped, and $\alpha > 0$. We also prove that there exists at least one positive solution when $g(0) < 0$ (semipositone), $g(s)$ is eventually positive for $s > 0$, and $0 < \alpha < 1$. We employ the method of sub-super solutions to prove our results.

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1. Introduction

In this paper, we consider the boundary value problem

$$-\Delta u = \phi g(u)u^{-\alpha} \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ϕ is a nonnegative function in $L^\infty(\Omega)$ such that $\phi > 0$ on some subset of Ω of positive measure, $g : [0, \infty) \rightarrow \mathbb{R}$ is continuous, and $\alpha > 0$.

We consider two cases. The first case assumes that $g(s)$ is positive and that $f(s) = s^{-\alpha}g(s)$ is roughly S-shaped in the sense that $\frac{s}{f(s)}$ rises from the origin to a maximum and then descends to a minimum before rising again. Alternatively, the geometry of the nonlinearity can be captured by bounding $g(s)$ above, then below, and then above again as s increases. This latter description is similar to that used in [9]. We prove the existence of three positive solutions. Our theorems generally extend results and methods for the nonsingular *positone* problems found in [1,2,7,9,11,14]. We also employ results and methods from the study of existence and uniqueness for singular problems where $g \equiv 1$. For example, see [10,13,12,18].

The second case assumes that $g(0) < 0$, g is eventually positive for $s > 0$, and $0 < \alpha < 1$. We prove the existence of at least one positive solution under suitable conditions. Our theorem generalizes the nonsingular *semipositone* results and methods in papers such as [3] and [4]. To obtain a positive subsolution we modify a construction found in [13]. We note that

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there have been a number of recent results regarding singular semipositone problems also referred to as *infinite semipositone* problems. For example, see [15] for the single equation Laplacian case and [16] for systems involving the p-Laplacian.

In Section 2 we consider the positone case and in Section 3 we consider the semipositone case. Our method of proof is to truncate the problem, apply the method of sub-super solutions to the truncated problem, and then return to the original problem via a limit.

2. The singular positone case: Multiplicity result

Theorem 2.1. *Suppose there exist constants a, b, c such that $0 < a < b < Mb < c$, where $M > 1$ will be defined later. Let $g : [0, \infty) \rightarrow (0, \infty)$ be a positive continuous function satisfying*

- (G1) $g(s) < (\frac{a}{m})^{\alpha+1}$ for $0 \leq s \leq a$,
- (G2) $g(s) > A^*b^{\alpha+1}$ for $b \leq s \leq Mb$, and
- (G3) $g(s) \leq (\frac{c}{m})^{\alpha+1}$ for $0 \leq s \leq c$,

where m and A^* will be defined later, ϕ is as described above and $\alpha > 0$. Then (1.1) has at least three positive solutions.

A simple example that satisfies the given hypotheses is given by the following.

Example 1. Consider $g(s) = \lambda e^{\frac{\beta s}{\beta+s}}$ where $\lambda > 0$ and $\beta > 0$ are parameters. Since g is monotonically increasing, to satisfy (G1)–(G2), it is enough to find a and b ($b > a$) such that

- $\lambda e^{\frac{\beta a}{\beta+a}} < (\frac{a}{m})^{\alpha+1}$ or $\lambda < (\frac{1}{m})^{\alpha+1} a^{\alpha+1} e^{\frac{-\beta a}{\beta+a}}$ and
- $\lambda e^{\frac{\beta b}{\beta+b}} > A^*b^{\alpha+1}$ or $\lambda > A^*b^{\alpha+1} e^{\frac{-\beta b}{\beta+b}}$.

One can visualize a and b as the critical points of $\frac{g(s)}{s}$, which are

$$\left[-1 + \frac{\beta}{2} \pm \frac{\sqrt{-4\beta + \beta^2}}{2} \right] \beta$$

so that $a = [-1 + \frac{\beta}{2} - \frac{\sqrt{-4\beta + \beta^2}}{2}] \beta$ and $b = [-1 + \frac{\beta}{2} + \frac{\sqrt{-4\beta + \beta^2}}{2}] \beta$. It is easy to see that

$$\lim_{\beta \rightarrow \infty} a = 1 \Rightarrow \frac{1}{m^{\alpha+1}} a^{\alpha+1} e^{\frac{-\beta a}{\beta+a}} \rightarrow \frac{1}{m^{\alpha+1}} \frac{1}{e}$$

and

$$b = O(\beta^2) \text{ as } \beta \rightarrow \infty \Rightarrow A^*b^{\alpha+1} e^{\frac{-\beta b}{\beta+b}} \rightarrow 0.$$

Therefore, we can choose β large enough so that

$$\Lambda_1 := A^*b^{\alpha+1} e^{\frac{-\beta b}{\beta+b}} < \frac{1}{m^{\alpha+1}} a^{\alpha+1} e^{\frac{-\beta a}{\beta+a}} := \Lambda_2.$$

Then (G1)–(G2) are satisfied for $\lambda \in (\Lambda_1, \Lambda_2)$. It is then trivial to satisfy (G3) for large c .

To prove Theorem 2.1, we will use the following three-solution theorem by Shivaji (see [17]). Note that by $u_1 < u_2$ we mean that $u_1 \leq u_2$ and $u_1 \neq u_2$.

Lemma 2.2 (Three-solution theorem). *Let \underline{u}_1 be a subsolution, \underline{u}_2 a strict subsolution, \bar{u}_1 a strict supersolution and \bar{u}_2 a supersolution of*

$$\begin{aligned} -\Delta u &= h(x, u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.2}$$

where h is a smooth function, such that $\underline{u}_1 < \bar{u}_1 < \bar{u}_2$, $\underline{u}_1 < \underline{u}_2 < \bar{u}_2$ and $\underline{u}_2 \not\leq \bar{u}_1$. Then (2.2) has at least three distinct solutions u_i , $i = 1, 2, 3$ such that $\underline{u}_1 \leq u_1 < u_2 < u_3 \leq \bar{u}_2$.

Remark 1. The proof of this lemma also shows that $u_3 \in [\underline{u}_1, \bar{u}_2] \setminus [\underline{u}_1, \bar{u}_1] \cup [\underline{u}_2, \bar{u}_2]$ and this fact is crucial in our analysis.

Remark 2. The smoothness condition of this lemma can be generalized to allow an L^p -Caratheodory function $h(x, u)$ with $p > N$. See [5] for details.

We study the following auxiliary problems to define the constants that appeared in the statement of the theorem as well as to find appropriate functions needed in the construction of sub and supersolutions.

2.1. Auxiliary problems

First consider the problem

$$\begin{aligned} -\Delta\psi &= A\phi h(\psi)\psi^{-\alpha} \quad \text{in } \Omega, \\ \psi &> 0 \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

where

$$h(s) = \begin{cases} 1, & s \geq 1, \\ 0, & s < 1. \end{cases}$$

In the next four lemmas we establish that there exists a smallest $A > 0$ for which (2.3) has a solution by using a standard sub-super solutions method and comparison principles.

Lemma 2.3. *There exists $A > 0$ such that (2.3) has a solution.*

Proof. Let $\Omega_1 \subset\subset \Omega$ be such that $|\Omega_1 \cap \{\phi > 0\}| > 0$ and let χ_{Ω_1} be the characteristic function on Ω_1 . Then the problem

$$\begin{aligned} -\Delta\psi &= \phi\psi^{-\alpha}\chi_{\Omega_1} \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.4}$$

has a positive solution, due to del Pino [6], say ψ . Let $d > 0$ be such that $d\psi \geq 1$ on Ω_1 . Then

$$-\Delta(d\psi) = d\phi\psi^{-\alpha}\chi_{\Omega_1} = d^{\alpha+1}\phi(d\psi)^{-\alpha}\chi_{\Omega_1} \leq d^{\alpha+1}\phi(d\psi)^{-\alpha}h(d\psi).$$

This shows that $\underline{\psi} = d\psi$ is a subsolution of (2.3) with $A = d^{\alpha+1}$. Now let $\bar{\psi}$ be the solution of

$$\begin{aligned} -\Delta\bar{\psi} &= d^{\alpha+1}\phi \quad \text{in } \Omega, \\ \bar{\psi} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then

$$-\Delta\bar{\psi} = d^{\alpha+1}\phi \geq d^{\alpha+1}\phi h(\bar{\psi})\bar{\psi}^{-\alpha}$$

holds in Ω and therefore $\bar{\psi}$ is a supersolution of (2.3). Moreover,

$$-\Delta\bar{\psi} = d^{\alpha+1}\phi \geq d^{\alpha+1}\phi h(\underline{\psi})\underline{\psi}^{-\alpha} \geq d^{\alpha+1}\phi\underline{\psi}^{-\alpha}\chi_{\Omega_1} = -\Delta\underline{\psi}.$$

Thus, by a standard comparison argument, we have $\bar{\psi} \geq \underline{\psi}$ in Ω . Since $\bar{\psi} = 0 = \underline{\psi}$ on $\partial\Omega$, it follows that (2.3) has a solution for $A = d^{\alpha+1}$. \square

Lemma 2.4. *If (2.3) has a solution for some $A_1 > 0$, then (2.3) has a solution for any $A \geq A_1$.*

Proof. Suppose (2.3) has a solution for some $A_1 > 0$, say ψ_{A_1} , and let $A_2 > A_1$. Then

$$-\Delta(\psi_{A_1}) = A_1\phi h(\psi_{A_1})\psi_{A_1}^{-\alpha} < A_2\phi h(\psi_{A_1})\psi_{A_1}^{-\alpha}.$$

Thus ψ_{A_1} is a subsolution of (2.3) corresponding to A_2 . It is easy to verify that a solution of $-\Delta(\bar{\psi}) = A_2\phi$ with zero boundary condition is a supersolution of (2.3). An argument similar to that in previous lemma shows that $\psi_{A_1} \leq \bar{\psi}$ in Ω and therefore (2.3) has a solution for A_2 , say ψ_{A_2} , such that $\psi_{A_1} \leq \psi_{A_2} \leq \bar{\psi}$. Since $A_2 > A_1$ is arbitrary, the lemma is proved. \square

Lemma 2.5. *There exists $A_1 > 0$ such that if (2.3) has a solution for some A , then $A \geq A_1$.*

Proof. Let y be the positive solution of $-\Delta y = \phi$ with zero boundary conditions and choose $A < \frac{1}{\|y\|_\infty}$. Let ψ_A be a positive solution of (2.3). Note that ψ_A is clearly nonnegative, and $-\Delta\psi_A = A\phi h(\psi_A)\psi_A^{-\alpha} \leq A\phi = -\Delta(Ay)$ and so, by a comparison argument, $\psi_A \leq Ay < 1$ in Ω . Thus $h(\psi_A) \equiv 0$, and so $-\Delta\psi_A = 0$. Hence we must have $\psi_A \equiv 0$. The lemma follows with $A_1 = \frac{1}{\|y\|_\infty}$. \square

Lemma 2.6. *Let $A^* = \inf\{A > 0: (2.3) \text{ has a solution}\}$. Then (2.3) has a solution for $A = A^*$.*

Proof. Let $A_n \searrow A^*$ and let $\psi_{n+1} \leq \psi_n$ be the corresponding sequence of solutions. By monotonicity, the pointwise limit $\psi_{A^*}(x) = \lim_{n \rightarrow \infty} \psi_n(x)$ exists and hence

$$A_n \phi h(\psi_n)(\psi_n)^{-\alpha} \rightarrow A^* \phi h(\psi_{A^*})(\psi_{A^*})^{-\alpha}$$

pointwise, where we have used the fact that h is continuous from the right. Since $A \phi h(\psi) \psi^{-\alpha}$ is uniformly bounded, we can apply standard regularity and imbedding theorems (see, for example, [8, Theorems 7.22, 9.11 and 9.15]), to derive a subsequence such that $\psi_n \rightarrow \psi_{A^*}$ in $C^{1,\gamma}$ for some $\gamma \in (0, 1)$ and ψ_{A^*} is a solution of (2.3) with $A = A^*$. \square

Let ψ_{A^*} be the solution of (2.3) with $A = A^*$ and let $M = \|\psi_{A^*}\|_\infty$. Note that $M \geq 1$. In fact $M > 1$. This follows from the fact that ψ_{A^*} is harmonic on the set $\{x: 0 < \psi_{A^*} < 1\}$, and that there must be a point $x_0 \in \{x: \psi_{A^*} = 1\}$ and a ball $B_\epsilon(x_1) \subset \{x: 0 < \psi_{A^*} < 1\}$ with $x_0 \in \partial B_\epsilon(x_1)$. By the Hopf maximum principle it follows that $\nabla \psi_{A^*}(x_0) \neq 0$ and so $\psi_{A^*}(x_0) = 1$ is not a local maximum.

Finally, consider the boundary value problem

$$\begin{aligned} -\Delta z &= \phi z^{-\alpha} \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.5}$$

where ϕ is as described earlier. It was shown by del Pino [6] that (2.5) has a unique positive solution $z \in C^{1,\gamma}(\Omega) \cap C^0(\overline{\Omega})$ for some $0 < \gamma < 1$. Define $m = \|z\|_\infty$.

We state and prove a simple lemma that is crucial in applying the three-solution theorem as well as in the later proof that the limiting solutions are indeed distinct.

Lemma 2.7 (Comparison lemma). *If there exists a constant $R > 0$ such that $-\Delta w = R\phi w^{-\alpha}$ in Ω and $-\Delta v \leq R\phi v^{-\alpha}$ in Ω with $v \leq w$ on $\partial\Omega$, then $v \leq w$ in Ω .*

Proof. Let $\Omega' = \{x: w < v\}$ so that $w = v$ on $\partial\Omega'$. Then

$$-\Delta v \leq R\phi v^{-\alpha} \leq R\phi w^{-\alpha} = -\Delta w \quad \text{in } \Omega'.$$

This gives $v \leq w$ in Ω' , a contradiction. \square

2.2. Truncated problem

Recall that $f(s) = g(s)s^{-\alpha}$ and let $\{\epsilon_n\}$ be a decreasing sequence of positive real numbers such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. For convenience we assume $\epsilon_1 < b$. Since $\lim_{s \rightarrow 0^+} f(s) = \infty$, without loss of generality, we can assume that $f(\epsilon_n) \leq f(\epsilon_{n+1})$. For each $n \in \mathbb{N}$, let

$$f_n(s) = \begin{cases} f(s), & s \geq \epsilon_n, \\ \min\{f(\epsilon_n), f(s)\}, & s < \epsilon_n. \end{cases}$$

For each $n \in \mathbb{N}$, consider the truncated problem

$$\begin{aligned} -\Delta u &= \phi f_n(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

Now we construct sub and supersolutions of the truncated problem, using functions from the auxiliary problems discussed earlier, satisfying the hypotheses of the three-solution theorem.

Lemma 2.8. $\underline{u}_1 = 0$ is a subsolution of (2.6).

Proof. Since $\phi f_n \geq 0$, we have that $-\Delta \underline{u}_1 = 0 \leq \phi f_n(\underline{u}_1)$ in Ω . This proves the lemma. \square

Lemma 2.9. $\underline{u}_2 = b\psi_{A^*}$ is a strict subsolution of (2.6).

Proof. Since ψ_{A^*} is zero on the boundary, $\partial\Omega$, we only need to show that

$$-\Delta \underline{u}_2 < \phi f_n(\underline{u}_2) \quad \text{in } \Omega.$$

Indeed,

$$-\Delta \underline{u}_2 = A^* b \phi h(\psi_{A^*})(\psi_{A^*})^{-\alpha} = A^* b^{\alpha+1} \phi h(\psi_{A^*})(b\psi_{A^*})^{-\alpha} = A^* b^{\alpha+1} \phi h(\underline{u}_2/b)(\underline{u}_2)^{-\alpha}.$$

If $\underline{u}_2/b < 1$, then $-\Delta \underline{u}_2 = 0 < \phi f_n(b\psi_{A^*})$. On the other hand, if $\underline{u}_2/b \geq 1$, then $h(\underline{u}_2/b) = 1$ and $b \leq \underline{u}_2 \leq Mb$. Since $\epsilon_n < b$ we have $f(s) = f_n(s)$ for $s \geq b$. Using (G2), we have

$$-\Delta \underline{u}_2 = \phi f_n(b\psi_{A^*})$$

as desired. Hence $-\Delta \underline{u}_2 < \phi f_n(\underline{u}_2)$ in Ω for all n . \square

Lemma 2.10. $\bar{u}_1 = \frac{a}{m}z$ is a strict supersolution of (2.6).

Proof. Using (G1), we get

$$-\Delta \bar{u}_1 = \phi \frac{a}{m}z^{-\alpha} = \phi \frac{a^{\alpha+1}}{m^{\alpha+1}} \left(\frac{a}{m}z\right)^{-\alpha} > \phi g\left(\frac{a}{m}z\right) \left(\frac{a}{m}z\right)^{-\alpha} = \phi f\left(\frac{a}{m}z\right) = \phi f(\bar{u}_1).$$

But $f \geq f_n$ for all $n \in \mathbb{N}$. Thus $-\Delta(\bar{u}_1) > \phi f_n(\bar{u}_1)$ as desired. \square

Lemma 2.11. $\bar{u}_2 = \frac{c}{m}z$ is a supersolution of (2.6).

Proof. The lemma follows using (G3) and an identical argument as in the proof of the previous lemma. \square

Now we verify the hypotheses of Lemma 2.2 to get three solutions of the truncated problem for each $n \in \mathbb{N}$. Clearly

$$\underline{u}_1 = 0 < \frac{a}{m}z = \bar{u}_1$$

and since $a < c$, we have

$$\bar{u}_1 = \frac{a}{m}z < \frac{c}{m}z = \bar{u}_2.$$

Clearly $\underline{u}_1 = 0 < b\psi_{A^*} = \underline{u}_2$. Next, we wish to show $\underline{u}_2 = b\psi_{A^*} < \frac{c}{m}z = \bar{u}_2$. We have

$$-\Delta \underline{u}_2 = -\Delta(b\psi_{A^*}) = bA^*\phi h(\psi_{A^*})(\psi_{A^*})^{-\alpha}.$$

If $\psi_{A^*} < 1$, then $-\Delta(b\psi_{A^*}) = 0 < \frac{a}{m}\phi z^{-\alpha} = -\Delta(\frac{a}{m}z)$. On the other hand, if $\psi_{A^*} \geq 1$ then

$$-\Delta(b\psi_{A^*}) = b^{\alpha+1}A^*\phi(b\psi_{A^*})^{-\alpha} \leq \phi \left(\frac{c}{m}\right)^{\alpha+1} (b\psi_{A^*})^{-\alpha}.$$

The last inequality holds since (G2) and (G3) together imply that we must have $b^{\alpha+1}A^* \leq (\frac{c}{m})^{\alpha+1}$. Then Lemma 2.7, together with the fact that $\|\underline{u}_2\| = Mb < c = \|\bar{u}_2\|$, implies that $\underline{u}_2 < \bar{u}_2$ as desired. Finally, since $\|\bar{u}_1\| = a < b \leq \|\underline{u}_2\|$, we must have $\bar{u}_1 \not\geq \underline{u}_2$.

Thus by Lemma 2.2, for each $n \in \mathbb{N}$, we have three solutions, u_{ni} , $i = 1, 2, 3$, of (2.6) where $0 < u_{n1} < \bar{u}_1$ and $\underline{u}_2 < u_{n2} \leq \bar{u}_2$. The third solution u_{n3} belongs to the set $[0, \bar{u}_2] \setminus ([0, \bar{u}_1] \cup [\underline{u}_2, \bar{u}_2])$.

2.3. Proof of the theorem

We will now prove that the solutions of the truncated problems converge to solutions of the original problem and that the limiting solutions are all distinct.

Lemma 2.12. Solutions of (2.6), $\{u_{ni}\}$ for $i = 1, 2, 3$, as obtained in previous section, converge to solutions of (1.1) in $C^1(\Omega) \cap C(\bar{\Omega})$.

Proof. Let $\delta > 0$ such that $g(s) \geq \delta \forall s \geq 0$ and $i \in \{1, 2, 3\}$ be fixed. Then $-\Delta u_{ni} \geq \delta \phi u_{ni}^{-\alpha}$ for every n , and so, by Lemma 2.7, $u_{ni} \geq w$ where w is the positive solution of

$$-\Delta w = \delta w^{-\alpha} \quad \text{in } \Omega,$$

$$w = 0 \quad \text{on } \partial\Omega.$$

Hence $w(x) \leq u_{ni}(x) \leq \bar{u}_2(x) \forall x \in \Omega$ and $\forall n$. It follows that $\{u_{ni}\}$ is uniformly bounded and is bounded below by the positive constant $\inf_{\Omega'} w$ on any compact subdomain Ω' of Ω . Hence $\{f_n(u_{ni}(x))\}$ is uniformly bounded on compact subdomains of Ω . Using standard regularity and embedding theorems (see [8], as cited before) we conclude that, without loss of generality, u_{ni} converges to some u_i in $C^{1+\mu}(\bar{\Omega}')$ for some $\mu > 0$. By a diagonalization argument we can say that, without loss of generality, this convergence holds on every $\Omega' \subset\subset \Omega$. Since $w = \bar{u}_2 = 0$ on the $\partial\Omega$ and $w \leq u_{ni} \leq \bar{u}_2$ we can also conclude that u_{ni} converges to u_i in $C(\bar{\Omega})$. It is now clear that u_i is a solution of (1.1). \square

Therefore we have three solutions, u_i , $i = 1, 2, 3$. Notice that $0 < u_1 < \frac{a}{m}z$; $b\psi_{A^*} < u_2 \leq \frac{c}{m}z$ and hence $u_1 \neq u_2$ since $\|\frac{a}{m}z\| = a < b \leq \|b\psi_{A^*}\|$. Thus the next two lemmas complete our proof.

Lemma 2.13. $u_3 \neq u_1$.

Proof. Suppose $u_3 = u_1$ so that $u_{n3} \rightarrow u_1$. Then $0 \leq u_{n3} \leq a$ for large n so that $g(u_{n3}) < (\frac{a}{m})^{\alpha+1}$ holds. Thus, for large n ,

$$-\Delta u_{n3} = \phi f_n(u_{n3}) \leq \phi g(u_{n3})(u_{n3})^{-\alpha} < \phi \left(\frac{a}{m}\right)^{\alpha+1} (u_{n3})^{-\alpha} = -\Delta \left(\frac{a}{m}z\right).$$

Thus by Lemma 2.7, $u_{n3} \leq \frac{a}{m}z$, a contradiction to the fact that u_{n3} must lie above \bar{u}_1 at some point. \square

Lemma 2.14. $u_3 \neq u_2$.

Proof. Suppose $u_3 = u_2$ so that $u_{n3} \rightarrow u_2$. Since $u_2 > \underline{u}_2$, for large n there exists $\epsilon > 0$ such that $u_{n3} \geq b + \epsilon$ on $\{x \in \Omega : \underline{u}_2 \geq b\}$. On the other hand, since u_{n3} must lie below \underline{u}_2 at some point, the set $\Omega_1 = \{x : \underline{u}_2 > u_{n3}\}$ is nonempty with $\underline{u}_2 = u_{n3}$ on $\partial\Omega_1$. If $x \in \Omega_1$ such that $\underline{u}_2 < b$, then $-\Delta u_{n3} \geq 0 = -\Delta \underline{u}_2$. If $x \in \Omega_1$ such that $\underline{u}_2 \geq b$, then $\epsilon_n < b \leq u_{n3}(x) < \underline{u}_2(x) \leq Mb$, so $f_n(u_{n3}) = f(u_{n3}) > A^*b^{1+\alpha}\underline{u}_2^{-\alpha}$, and it follows that $-\Delta u_{n3} > -\Delta \underline{u}_2$. Therefore, $\underline{u}_2 \leq u_{n3}$ on Ω_1 , a contradiction. \square

3. The singular semipositone case: Existence result

In this section we use similar methods to prove the existence of at least one positive solution for the problem

$$\begin{aligned} -\Delta u &= \phi g(u)u^{-\alpha} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.7}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $0 < \alpha < 1$, ϕ is a bounded nonnegative function which is positive on a subset of positive measure, and $g : [0, \infty) \rightarrow \mathbb{R}$ is a bounded continuous function such that $g(0) < 0$.

Since g is bounded and $g(0) < 0$ it is trivial to find a positive constant $C \geq \max_{s \geq 0} g(s)s^{-\alpha}$. Let \bar{u} be the solution of

$$\begin{aligned} -\Delta u &= \phi C \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

It is clear that \bar{u} serves as a supersolution for (3.7) and that, by a simple comparison argument, \bar{u} will lie above any solution or subsolution of (3.7). Moreover, it will be clear that \bar{u} will play a similar role for all of the truncated problems described below.

The nontrivial part of the argument is to identify a positive subsolution. To do this we combine ideas and computations from [7] and [13]. In brief we will use $\underline{u} = w^\lambda$, where $\lambda = \frac{2}{\alpha+1} > 1$ and where w is the positive solution of

$$\begin{aligned} -\Delta w &= \phi \chi_{\Omega'} \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.9}$$

where $\Omega' \subset\subset \Omega$ and $\chi_{\Omega'}$ is the standard characteristic function. In order to guarantee $w > 0$ in Ω we need to have $\phi > 0$ on a subset of Ω' with positive measure. This will be clearly satisfied in the context below. Problems of this sort are investigated in some depth in [7] where it is shown that a careful study of (3.9) can lead to sharp bounds for existence theorems. The technical difficulty in the present case lies in obtaining a positive lower bound on $|\nabla w|$ in the region $\Omega \setminus \Omega'$ where w is harmonic. In [13] a similar subsolution construction was used for a different type of singular problem. In that case the authors chose w to be a multiple of the principal eigenfunction of $-\Delta$ with zero Dirichlet boundary condition.

Let $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) > \epsilon\}$, where $d(x, \partial\Omega)$ represents the distance from the point x to the boundary $\partial\Omega$. Let w_ϵ be the positive solution of (3.9) with $\Omega' = \Omega_\epsilon$. Since $\phi \chi_{\Omega_\epsilon}$ converges to $\phi \chi_\Omega$ in $L^p(\Omega)$ for any $p \in [1, \infty)$ it follows that, without loss of generality, $w_\epsilon \rightarrow w_0$ in $C^1(\bar{\Omega})$, where w_0 is the positive solution of

$$\begin{aligned} -\Delta w_0 &= \phi \quad \text{in } \Omega, \\ w_0 &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.10}$$

Let $\delta := \min\{|\nabla w_0(x)| : x \in \partial\Omega\} > 0$. Let $\epsilon > 0$ such that $\min\{|\nabla w_\epsilon(x)| : x \in \bar{\Omega} \setminus \Omega_\epsilon\} > \frac{\delta}{2}$. Let $d := \min\{w_\epsilon(x) : x \in \bar{\Omega}_\epsilon\}$ and let $D := \max\{w_\epsilon(x) : x \in \bar{\Omega}_\epsilon\}$.

Note that a simple estimate yields $w_\epsilon \geq \frac{\delta}{2}\epsilon$ on $\partial\Omega_\epsilon$ and it follows by the maximum principle that this estimate extends to all of Ω_ϵ so $d \geq \frac{\delta}{2}\epsilon$.

Now we are prepared to state the existence theorem.

Theorem 3.1. Assume that $g : [0, \infty) \rightarrow \mathbb{R}$ is a bounded continuous function with $g(0) < 0$ and $0 < \alpha < 1$, and let $\lambda = \frac{2}{\alpha+1}$. If $g(s^\lambda) \geq -\frac{\lambda(\lambda-1)\delta^2}{2\|\phi\|_\infty} := -K$ on $[0, d]$, and if $g(s^\lambda) \geq \lambda s$ on $[d, D]$, then (3.7) has at least one positive solution.

Example 2. Consider $g(s) = [e^{\frac{\beta s}{1+s}} - (K + 1)]$, where $\beta > 0$ is a parameter. Then $g(0) = -K$, and, since g is monotone increasing, we have that $g(s) \geq -K$ on $[0, d]$. Now choose β large enough so that $e^{\frac{\beta d^\lambda}{1+d^\lambda}} \geq \lambda D + K + 1$. This guarantees that $g(s^\lambda) \geq \lambda s$ on $[d, D]$. Therefore, the hypotheses of Theorem 3.1 are satisfied for β large.

Proof. Let $f(s) := g(s)s^{-\alpha}$ and for each $n \in \mathbb{N}$, define

$$f_n(s) := \begin{cases} f(s), & s \geq \epsilon_n, \\ \max\{f(\epsilon_n), f(s)\}, & 0 < s \leq \epsilon_n, \end{cases}$$

where $\{\epsilon_n\}$ is a decreasing sequence of positive real numbers such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Note $\lim_{s \rightarrow 0^+} f(s) = -\infty$, so each $f_n \equiv f(\epsilon_n)$ on some interval $(0, \epsilon'_n)$, and so f_n can clearly be extended to have the value $f(\epsilon_n)$ at 0.

We will show that the truncated boundary value problem

$$\begin{aligned} -\Delta u &= \phi f_n(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.11}$$

has a positive solution by constructing a positive subsolution and an appropriately ordered supersolution. For notational convenience we let $\chi_{[a,b]}$ denote the characteristic function over $\{x: a \leq w_\epsilon \leq b\}$, where $\epsilon > 0$ is as described earlier and fixed. Then $\underline{u} := w_\epsilon^\lambda$ satisfies

$$\begin{aligned} -\Delta(w_\epsilon^\lambda) &= -\lambda(\lambda - 1)w_\epsilon^{\lambda-2}|\nabla w_\epsilon|^2 + \lambda\phi w_\epsilon^{\lambda-1}\chi_{\Omega_\epsilon} \\ &\leq -\lambda(\lambda - 1)\frac{\delta^2}{4}\frac{\phi}{\|\phi\|_\infty}(w_\epsilon^\lambda)^{-\alpha}\chi_{[0,d]} + \lambda\phi w_\epsilon(w_\epsilon^\lambda)^{-\alpha}\chi_{[d,D]} \\ &= \phi \left[-\lambda(\lambda - 1)\frac{\delta^2}{4}\frac{1}{\|\phi\|_\infty}\chi_{[0,d]} + \lambda w_\epsilon \chi_{[d,D]} \right] (w_\epsilon^\lambda)^{-\alpha} \\ &\leq \phi g(w_\epsilon^\lambda)(w_\epsilon^\lambda)^{-\alpha} \end{aligned}$$

for each $n \in \mathbb{N}$.

Thus $\underline{u} := w_\epsilon^\lambda$ is a subsolution of (3.11) for each n . Also \bar{u} , solution of (3.8), is a supersolution of (3.11) with $\underline{u} \leq \bar{u}$. For each $n \in \mathbb{N}$, let u_n be a solution of (3.11) lying in the order interval $[\underline{u}, \bar{u}]$.

Note that on compact subdomains of Ω we have an L^∞ bound on $\{u_n\}$ and we have a strictly positive lower bound. Standard regularity arguments show that, without loss of generality, u_n converges to a positive solution u of (3.7) in $C^1(\Omega) \cap C(\bar{\Omega})$. \square

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