ACTIONS OF FINITE DIMENSIONAL NON-COMMUTATIVE, NON-COCOMMUTATIVE HOPF ALGEBRAS ON RINGS

By

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Abstract

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Thesis under the direction of James Kuzmanovich, Ph.D.

In 1954, Shephard and Todd [20] showed that if A is a polynomial ring and G is a finite group acting as automorphisms on A, then the ring of invariants $A^G = \{a \in A \mid g \cdot a = a, \forall g \in G\}$ is again a polynomial ring exactly when G is generated by reflections. The major goal of this thesis is the computation of several examples en route to a conjecture for an analogous result regarding the ring of invariants for some class of "nice" algebras under finite dimensional Hopf algebra actions.

We begin with an introduction to the general study of Hopf algebras and their basic properties, then explain why they are a natural choice to generalize the action of finite groups on rings. We then show that in order to generalize existing theories, we must consider actions of "nontrivial" Hopf algebras, in particular, those that are not isomorphic to group rings or their duals. We compute several examples of such actions, and in particular, we prove that there are no actions of nontrivial semisimple Hopf algebras with dimension less than or equal to 15 on polynomial algebras.

Chapter 1: Algebras and coalgebras

Much of the material in the introductory chapters, 1 and 2, covering basic constructions, results, and examples of algebras, coalgebras, bialgebras and Hopf algebras, is an amalgam of material found in Dăscălescu, et. al. [4], Montgomery [18], and Sweedler [21]. In many cases, we expand on these texts, either providing simpler proofs which emphasize only what is necessary for the purpose of this thesis, or give explicit calculation and verification for much of the "folklore" of the subject.

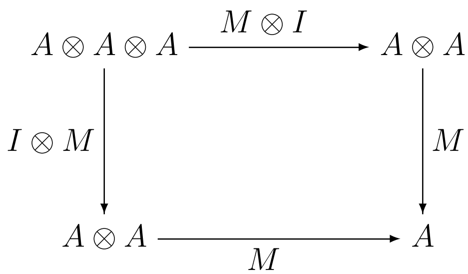
1.1 Definitions and examples

Let k be a field. All vector spaces and tensor products are assumed to be over k. Throughout we will assume the reader to be familiar with concepts of the tensor product of two vector spaces and many properties of algebras over the field k. At certain points, we will restrict our discussion to algebraically closed fields of characteristic zero, and in fact, simply to $\mathbb{C}$. We begin by presenting the traditional definition of a k-algebra.

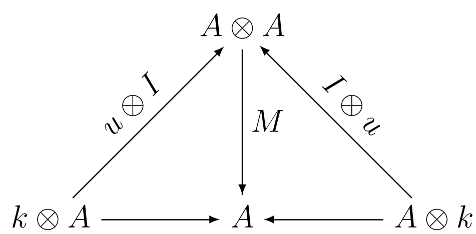
Definition 1.1.1. *A unitary ring A becomes a k-algebra with the structure map $\phi : k \to A$, a unitary ring morphism, such that the image of ϕ is contained in the center of A.*

In order to dualize the notion of an algebra, we give an equivalent definition using commutative diagrams.

Definition 1.1.2. *A k-**algebra** is a triple (A, M, u) where A is a k-vector space, and $M : A \otimes A \to A$, $u : k \to A$ are k-linear maps such that the following diagrams are* *commutative:*



(1.1)



(1.2)

We have used I to denote the identity map on A, while the unadorned arrows are the canonical isomorphisms given by scalar multiplication. The maps M and u are called the **multiplication** *and* **unit***, respectively, of the algebra A.*

Remark 1.1.3. Note that diagram (1.1) ensures that the multiplication of A is associative. Also, using diagram (1.2), if $\lambda \in k$ such that $u(\lambda) = 0$, then for any $a \in A$, we have

$$M \circ (u \otimes I)(\lambda \otimes a) = M(0 \otimes a) = 0.$$

Thus, by the commutativity of (1.2), $\lambda a = 0$ for all $a \in A$ and therefore u is injective. Also, since

$$M \circ (u \otimes I)(1_k \otimes a) = u(1_k)a = a, \text{ and}$$
$$M \circ (I \otimes u)(a \otimes 1_k) = au(1_k) = a,$$

we see that $u(1_k) = 1_A$. Note that we have shown the statement $u(1_k) = 1_A$ is equivalent to the commutativity of (1.2) and hence u is an embedding of the field k into A. Thus, $u(k) \subseteq Z(A)$.

Therefore, Definitions 1.1.1 and 1.1.2 are equivalent. The map M endows A with the structure of a unitary ring with $1_A = u(1_k)$. The role of the map ϕ is simply played by u. Conversely, if we have a unitary ring with structure map ϕ, we define the map M by $x \otimes y \mapsto xy$ and let $u = \phi$. ■

Now, simply by reversing the arrows in Definition 1.1.2, we obtain the dual structure of a k-coalgebra.

Definition 1.1.4. *A k-**coalgebra** is a triple (C, Δ, ε) where C is a k-vector space, and $\Delta : C \to C \otimes C$, $\varepsilon : C \to k$ are morphisms of k-vector spaces such that the following diagrams are commutative:*

$$\begin{array}{ccc} C & \xrightarrow{\quad\Delta\quad} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle I \otimes \Delta} \\ C \otimes C & \xrightarrow[\Delta \otimes I]{} & C \otimes C \otimes C \end{array} \tag{1.3}$$

$$\begin{array}{ccccc} k \otimes C & \xleftarrow{1\otimes} & C & \xrightarrow{\otimes 1} & C \otimes k \\ & {\scriptstyle\varepsilon \otimes I}\nwarrow & \downarrow{\scriptstyle\Delta} & \nearrow{\scriptstyle I \otimes \varepsilon} & \\ & & C \otimes C & & \end{array} \tag{1.4}$$

The maps Δ and ε are called the **comultiplication** *and* **counit**, *respectively, of the coalgebra C. Alternatively, we sometimes refer to the counit as an* **augmentation map**. *We call the commutativity of diagram* (1.3) **coassociativity**.

Now we give several examples of algebras and coalgebras, the second of which, the group algebra, will be our prototypical example throughout this text.

Example 1.1.5 (The Group-like Coalgebra)**.** Let S be a nonempty set. Denote by kS the vector space formed using S as a basis. Then kS becomes a coalgebra via

$$\Delta(s) = s \otimes s, \text{ and } \varepsilon(s) = 1$$

for all $s \in S$. First, we check the coassociativity:

$$(I \otimes \Delta) \circ \Delta(s) = s \otimes \Delta(s) = s \otimes (s \otimes s) = (s \otimes s) \otimes s = \Delta(s) \otimes s = (\Delta \otimes I) \circ \Delta(s).$$

The counit property is easily verified as well. ■

This shows that *any* k-vector space can be endowed with a coalgebra structure. Now, we introduce a very important example.

Example 1.1.6 (The Group Algebra)**.** Let G be a multiplicative group. Denote by kG the **group algebra** which has G as a basis for its vector space. The multiplication of kG is the obvious one, induced by the multiplication of G and extended linearly. The unit map is given by $u(\alpha) = \alpha 1_G$, for all $\alpha \in k$. Additionally, we can endow kG with the structure of a coalgebra by using the preceding example, letting $S = G$. Later, we will be very interested in objects, such as kG, which can be simultaneously endowed with the structure of an algebra and coalgebra. ■

Example 1.1.7 (The Trigonometric Coalgebra)**.** Let C be the k-vector space with basis $\{s, c\}$. Then define comultiplication Δ and counit ε by

$$\begin{aligned} \Delta(s) &= s \otimes c + c \otimes s, \quad \varepsilon(s) = 0 \\ \Delta(c) &= c \otimes c - s \otimes s, \quad \varepsilon(c) = 1. \end{aligned}$$

This forms a coalgebra, with a resemblance to the angle sum formulas. Since we may extend our results linearly by the basis elements, we must only check that the comultiplication is coassociative on s and c. Further, we must check that diagram

(1.4), the counit property, holds for both s and c. Below are the details for c:

$$\begin{aligned}
&(I \otimes \Delta) \circ \Delta(c) \\
=&(I \otimes \Delta) \circ (c \otimes c - s \otimes s) \\
=&c \otimes (c \otimes c - s \otimes s) - s \otimes (s \otimes c + c \otimes s) \\
=&c \otimes c \otimes c - s \otimes s \otimes c - s \otimes c \otimes s - c \otimes s \otimes s \\
=&(c \otimes c - s \otimes s) \otimes c - (s \otimes c + c \otimes s) \otimes s \\
=&\Delta(c) \otimes c - \Delta(s) \otimes s \\
=&(\Delta \otimes I) \circ \Delta(c).
\end{aligned}$$

Now we check the counit property; that is,

$$\begin{aligned}
&(\varepsilon \otimes I) \circ \Delta(c) = \varepsilon(c) \otimes c - \varepsilon(s) \otimes s = 1 \otimes c, \text{ and} \\
&(I \otimes \varepsilon) \circ \Delta(c) = c \otimes \varepsilon(c) - s \otimes \varepsilon(s) = c \otimes 1.
\end{aligned}$$

The computations for s are similar. ■

Example 1.1.8. The field k is a coalgebra with comultiplication defined by $\Delta(\lambda) = \lambda \otimes 1$, the canonical isomorphism, and counit $\varepsilon = I$. Note that this is a special case of Example 1.1.5 when S is a set of only one element. ■

Before introducing the next examples, we need to define a linear map which will be utilized throughout.

Definition 1.1.9. *Let V and W be k-vector spaces. The k-linear map $T : V \otimes W \to W \otimes V$ defined by $v \otimes w \mapsto w \otimes v$, is called the* **twist map***.*

Example 1.1.10 (The Opposite Algebra)**.** Let (A, M, u) be an algebra. The **opposite algebra** (A^{op}, M^{op}, u) has A as its underlying k-vector space but the twisted multiplication:

$$M^{op}(x \otimes y) = (M \circ T)(x \otimes y).$$

It is not difficult to check A^{op} is an algebra. ■

Example 1.1.11 (The Coopposite Coalgebra)**.** Let (C, Δ, ε) be a coalgebra. The **coopposite coalgebra** $(C^{cop}, \Delta^{cop}, \varepsilon)$ has C as its underlying k-vector space but the twisted comultiplication:

$$\Delta^{cop}(c) = (T \circ \Delta)(c).$$

Again, the verification is straightforward. ■

Example 1.1.12 (The Tensor Product of Coalgebras)**.** Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. We claim that the tensor product, $C \otimes D$ becomes a coalgebra by the following maps:

$$\Delta(c \otimes d) = (I \otimes T \otimes I) \circ (\Delta_C \otimes \Delta_D)(c \otimes d), \text{ and } \varepsilon(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d).$$

We postpone the verification of this example for the next section since it motivates a method of simplification for the unwieldy calculations in a coalgebra. ■

1.2 Computation in coalgebras, sigma notation

It is clear that computations in a coalgebra will be more difficult than those in an algebra. When the comultiplication is applied to a single element, it expands into a finite sum of pairs of elements. In the case of algebras, we may expand by induction the notion of associativity to as many multiplicands as possible, so that the multiplication produces unique results. It is natural to inquire whether or not an analogous property holds for coalgebras. To investigate this, let (C, Δ, ε) be a coalgebra and recursively define the sequence of maps, $\{\Delta_n\}_{n \geq 1}$, by:

$$\Delta_1 = \Delta, \quad \Delta_n : C \to C \otimes \cdots \otimes C \quad \text{(with } C \text{ appearing } n+1 \text{ times)},$$

$$\Delta_n = (\Delta \otimes I^{n-1}) \circ \Delta_{n-1}, \text{ for } n \geq 2.$$

The following proposition dualizes the desired notion to coalgebras. We will call this property **generalized coassociativity**.

Proposition 1.2.1. *Let* (C, Δ, ε) *be a coalgebra. Then for any* $n \geq 2$*,* $j \in \{0, 1, \ldots, n-1\}$*, the following equality holds:*

$$\Delta_n = (I^j \otimes \Delta \otimes I^{n-1-j}) \circ \Delta_{n-1}.$$

Proof. The proof is by induction on n. Notice that for $n = 2$ we must show $(I \otimes \Delta) \circ \Delta = (\Delta \otimes I) \circ \Delta$, but this is just the commutativity of diagram (1.3), that is, coassociativity. Now, we assume the equation holds for n and let $j \in \{1, \ldots, n\}$. Then,

$$\begin{aligned}
&(I^j \otimes \Delta \otimes I^{n-j}) \circ \Delta_n \\
=&(I^j \otimes \Delta \otimes I^{n-j}) \circ (I^{j-1} \otimes \Delta \otimes I^{n-j}) \circ \Delta_{n-1} \\
=&(I^{j-1} \otimes ((I \otimes \Delta) \circ \Delta) \otimes I^{n-j}) \circ \Delta_{n-1} \\
=&(I^{j-1} \otimes ((\Delta \otimes I) \circ \Delta) \otimes I^{n-j}) \circ \Delta_{n-1} \\
=&(I^{j-1} \otimes \Delta \otimes I^{n-j+1}) \circ (I^{j-1} \otimes \Delta \otimes I^{n-j}) \circ \Delta_{n-1} \\
=&(I^{j-1} \otimes \Delta \otimes I^{n-j+1}) \circ \Delta_n.
\end{aligned}$$

Where we have again made use of the coassociativity in the fourth line above. So we have "shifted" the comultiplication by one place. Notice that if $j = 0$ then $\Delta_n = (I^j \otimes \Delta \otimes I^{n-1-j}) \circ \Delta_{n-1}$, by definition. The result follows by induction on j. ■

In practice, this tells us that if we are to apply Δ to a c_i in the string $\Delta_{n-1}(c) = c_1 \otimes c_2 \otimes \cdots \otimes c_n$ in order to obtain Δ_n, it is irrelevant which term we choose to expand, our answer is always unique. This fact allows us to make use of a very helpful notation, introduced by Heyneman and Sweedler [7], when performing calculations in a coalgebra.

Notation 1.2.2. Let (C, Δ, ε) be a coalgebra. In general, we could write the comultiplication applied to the element $c \in C$ as

$$\Delta(c) = \sum_i c_{i1} \otimes c_{i2}.$$

Because we wish to emphasize mostly the form of the element $\Delta(c) \in C \otimes C$, we simply suppress the subscript 'i' and denote for any $c \in C$

$$\Delta(c) = \sum c_1 \otimes c_2.$$

We call this convention the **sigma notation**. Furthermore, generalized coassociativity allows us to write unambiguously for any $c \in C$,

$$\Delta_n(c) = \sum c_1 \otimes \cdots \otimes c_{n+1}.$$

We now present some formulas making use of the sigma notation to aid in our calculations.

Facts 1.2.3. *Let (C, Δ, ε) be a coalgebra and $c \in C$. The following are reformulations of the coassociativity and counit properties.*

(*i*) (Coassociativity) $\Delta_2(c) = \sum \Delta(c_1) \otimes c_2 = \sum c_1 \otimes \Delta(c_2) = \sum c_1 \otimes c_2 \otimes c_3$

(*ii*) (Counit) $\sum \varepsilon(c_1)c_2 = \sum c_1\varepsilon(c_2) = c.$

Proof. (*i*) The first assertion is clear by chasing through diagram (1.3) in the definition of a coalgebra. We have already agreed to write the unique element in $C \otimes C \otimes C$ given by the first two expressions in (*i*) as $\sum c_1 \otimes c_2 \otimes c_3$. Alternatively, we could have written:

$$\Delta_2(c) = \sum (c_1)_1 \otimes (c_1)_2 \otimes c_2 = \sum c_1 \otimes (c_2)_1 \otimes (c_2)_2 = \sum c_1 \otimes c_2 \otimes c_3.$$

(ii) We may justify this formulation for the counit property by first writing the canonical isomorphisms, $\phi_L : k \otimes C \to C$, and $\phi_R : C \otimes k \to C$, given by scalar multiplication. Then, diagram (1.4) may be written as:

$$I = \phi_L \circ (\varepsilon \otimes I) \circ \Delta = \phi_R \circ (I \otimes \varepsilon) \circ \Delta,$$

and so, $c = \sum \varepsilon(c_1)c_2 = \sum c_1\varepsilon(c_2)$. ■

We now return to the concept of the tensor product of two coalgebras to illustrate the power of this notation.

Proposition 1.2.4. *Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. Then $(C \otimes D, \Delta, \varepsilon)$ is a coalgebra with the following comultiplication and counit maps:*

$$\Delta(c \otimes d) = (I \otimes T \otimes I) \circ (\Delta_C \otimes \Delta_D)(c \otimes d)$$
$$\varepsilon(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d).$$

Proof. First, we note that we may rewrite the comultiplication Δ in terms of the sigma notation as $\Delta(c \otimes d) = \sum c_1 \otimes d_1 \otimes c_2 \otimes d_2$. Then we have that,

$$\begin{aligned} &(\Delta \otimes I_{C\otimes D}) \circ \Delta(c \otimes d) \\ =&(\Delta \otimes I_{C\otimes D})(\sum c_1 \otimes d_1 \otimes c_2 \otimes d_2) \\ =&\sum (c_1)_1 \otimes (d_1)_1 \otimes (c_1)_2 \otimes (d_1)_2 \otimes c_2 \otimes d_2 \\ =&\sum c_1 \otimes d_1 \otimes c_2 \otimes d_2 \otimes c_3 \otimes d_3, \end{aligned}$$

and also,

$$\begin{aligned} &(I_{C\otimes D} \otimes \Delta) \circ \Delta(c \otimes d) \\ =&(I_{C\otimes D} \otimes \Delta)(\sum c_1 \otimes d_1 \otimes c_2 \otimes d_2) \\ =&\sum c_1 \otimes d_1 \otimes (c_2)_1 \otimes (d_2)_1 \otimes (c_2)_2 \otimes (d_2)_2 \\ =&\sum c_1 \otimes d_1 \otimes c_2 \otimes d_2 \otimes c_3 \otimes d_3, \end{aligned}$$

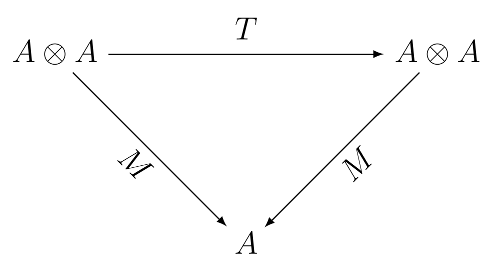
showing that the comultiplication is coassociative. We use the reformulation of the counit property above and notice that

$$\begin{aligned} &\sum \varepsilon(c_1 \otimes d_1)(c_2 \otimes d_2) \\ =&\sum \varepsilon_C(c_1)\varepsilon_D(d_1)(c_2 \otimes d_2) \\ =&\left(\sum \varepsilon_C(c_1)c_2\right) \otimes \left(\sum \varepsilon_D(d_1)d_2\right) \\ =&c \otimes d. \end{aligned}$$

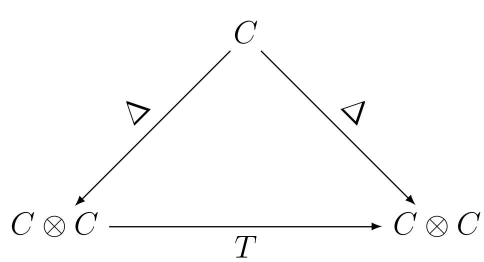
Similarly, $\sum(c_1 \otimes d_1)\varepsilon(c_2 \otimes d_2) = c \otimes d$, and hence, $(C \otimes D, \Delta, \varepsilon)$ is a coalgebra. ■

We conclude this section with two important categorical definitions. The first is equivalent to the usual definition for commutativity; the second is its dual.

Definition 1.2.5. *(i) An algebra (A, M, u) is said to be* **commutative** *if the following diagram commutes:*



(ii) A coalgebra (C, Δ, ε) is said to be **cocommutative** *if the following diagram commutes:*

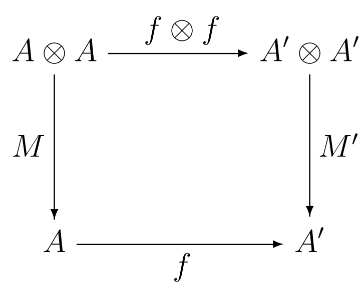


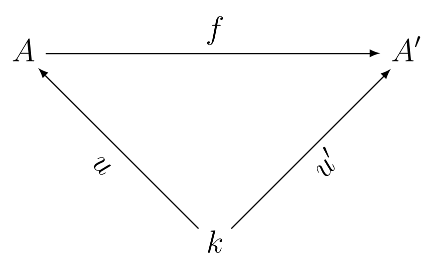
In sigma notation, we may write this as $\sum c_1 \otimes c_2 = \sum c_2 \otimes c_1$ for any $c \in C$.

1.3 Morphisms, coideals, quotient structures

In this section we verify that all of the usual constructions for algebraic structures are valid in the case of coalgebras. We define the maps between algebras the usual way, but use commutative diagrams for the purpose of dualizing the definitions. We then proceed to construct the "ideals" for coalgebras; that is, the objects that will lead to natural quotient structures.

Definition 1.3.1. *Let* (A, M, u) *and* (A', M', u') *be* k*-algebras. A* k*-linear map* $f : A \to A'$ *is called an* **algebra morphism** *if the following diagrams are commutative:*

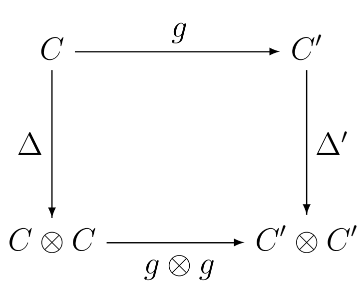


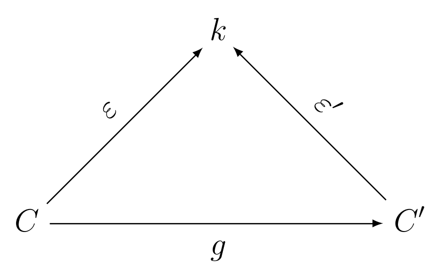


Alternatively, we could write for any $a, b \in A$, $\lambda \in k$, *that* $f(ab) = f(a)f(b)$ *and* $f(u(\lambda)) = u'(\lambda)$, *or equivalently,* $f(1_A) = 1_{A'}$.

Dualizing this definition to the case of coalgebras we have:

Definition 1.3.2. *Let* (C, Δ, ε) *and* $(C', \Delta', \varepsilon')$ *be* k*-coalgebras. A* k*-linear map* $g : C \to C'$ *is called a* **coalgebra morphism** *if the following diagrams are commutative:*





In sigma notation, we could write for any $c \in C$,

$$\Delta'(g(c)) = \sum g(c)_1 \otimes g(c)_2 = \sum g(c_1) \otimes g(c_2), \text{ and } \varepsilon(c) = \varepsilon'(g(c)).$$

Definition 1.3.3. *Let* (C, Δ, ε) *be a coalgebra. A subspace* $J \subseteq C$ *is called a*

(i) **co-subalgebra** *if* $\Delta(J) \subseteq J \otimes J$,

(ii) **coideal** *if* $\Delta(J) \subseteq J \otimes C + C \otimes J$ *and* $\varepsilon(J) = 0$.

Here we extend some important results for algebras to the case of coalgebras.

Proposition 1.3.4. *Let* $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$ *be coalgebras and* $f : C \to D$ *a coalgebra morphism. Then,*

(i) $f(C)$ *is a co-subalgebra of* D,

(ii) $\mathrm{Ker}(f)$ *is a coideal of* C.

Proof. (i) $f(C)$ is a subspace of D since f is k-linear. Using the fact that f is a coalgebra morphism, we have that

$$\begin{aligned}\Delta_D(f(C)) =& (f \otimes f)\Delta_C(C) \\ \subseteq& (f \otimes f)(C \otimes C) \\ =& f(C) \otimes f(C),\end{aligned}$$

showing $f(C)$ is a co-subalgebra.

(ii) Again, we know that $\mathrm{Ker}(f)$ is certainly a subspace of C. Let $\{v_\alpha\}_{\alpha \in \mathcal{A}_1}$ be a basis for $\mathrm{Ker}(f)$ which we complete with the linearly independent set, $\{v_\alpha\}_{\alpha \in \mathcal{A}_2}$ so that $\{v_\alpha\}_{\alpha \in \mathcal{A}_1 \cup \mathcal{A}_2}$ is a basis for all of C. Then $\{f(v_\alpha)\}_{\alpha \in \mathcal{A}_2}$ is a linearly independent set in D. Note that since f is a coalgebra morphism, $0 = \Delta_D(f(\mathrm{Ker}\, f)) = (f \otimes f)\Delta_C(\mathrm{Ker}\, f)$. Thus,

$$\Delta_C(\mathrm{Ker}\, f) \subseteq \mathrm{Ker}(f \otimes f). \tag{1.5}$$

Now, take $x \in \mathrm{Ker}(f \otimes f)$. That is,

$$x = \sum_{\alpha, \beta \in \mathcal{A}_1 \cup \mathcal{A}_2} \lambda_{\alpha\beta} v_\alpha \otimes v_\beta,$$

so that after applying $f \otimes f$, we have

$$\sum_{\alpha,\beta \in \mathcal{A}_1 \cup \mathcal{A}_2} \lambda_{\alpha\beta} f(v_\alpha) \otimes f(v_\beta) = 0.$$

Now, for each $\alpha \in \mathcal{A}_1$ we have that $f(v_\alpha) = 0$. So we may rewrite the above sum (less redundantly) as

$$\sum_{\alpha,\beta \in \mathcal{A}_2} \lambda_{\alpha\beta} f(v_\alpha) \otimes f(v_\beta) = 0.$$

Then by the linearly independence of the set $\{f(v_\alpha) \otimes f(v_\beta)\}_{\alpha,\beta \in \mathcal{A}_2}$, (indeed, it forms a basis for $f(C) \otimes f(C)$), we have that $0 = \lambda_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{A}_2$. Thus, we can rewrite

$$x = \sum_{\beta \in \mathcal{A}_1 \cup \mathcal{A}_2} \sum_{\alpha \in \mathcal{A}_1} \lambda_{\alpha\beta} v_\alpha \otimes v_\beta + \sum_{\alpha \in \mathcal{A}_1 \cup \mathcal{A}_2} \sum_{\beta \in \mathcal{A}_1} \lambda_{\alpha\beta} v_\alpha \otimes v_\beta,$$

which implies that

$$x \in \operatorname{Ker}(f) \otimes C + C \otimes \operatorname{Ker}(f). \tag{1.6}$$

Combining (1.5) and (1.6) gives $\Delta_C(\operatorname{Ker}(f)) \subseteq \operatorname{Ker}(f \otimes f) \subseteq \operatorname{Ker}(f) \otimes C + C \otimes \operatorname{Ker}(f)$. Moreover, if $x \in \operatorname{Ker}(f)$, then $0 = \varepsilon_D(f(x)) = \varepsilon_C(x)$, so $\varepsilon_C(\operatorname{Ker}(f)) = 0$. Therefore, $\operatorname{Ker}(f)$ is a coideal. ■

Now we can introduce quotient structures for coalgebras. Let (C, Δ, ε) be a coalgebra and J be a coideal. Define the map $\pi : C \to C/J$ to be the canonical vector space projection. We will write $\pi(c) = \overline{c}$ as the coset of c modulo J. We must first recall a property of quotient vector spaces.

Lemma 1.3.5 (Universal Property of Quotient Vector Spaces)**.** *Let V be a k-vector space, and W a subspace of V. Whenever W' is a k-vector space and $\psi : V \to W'$ a linear map such that $W \subseteq \operatorname{Ker}(\psi)$ there exists a unique linear map $\phi : V/W \to W'$ such that $\psi = \phi \circ \pi$.*

Proof. Define the map $\phi : V/W \to W'$ by $\phi(\overline{v}) = \psi(v)$, that is, $\psi = \phi \circ \pi$. The map, ϕ, is well defined since whenever $v_1, v_2 \in V$ are representatives of the coset $\overline{v} \in V/W$, then $v_1 = v_2 + w$ for some $w \in W$. Hence,

$$\psi(v_1) = \psi(v_2 + w) = \psi(v_2) + \psi(w) = \psi(v_2).$$

Moreover, ϕ is linear, since

$$\phi(\overline{v_1} + \overline{v_2}) = \phi(\overline{v_1 + v_2}) = \psi(v_1 + v_2) = \psi(v_1) + \psi(v_2) = \phi(\overline{v_1}) + \phi(\overline{v_2}).$$

Finally, ϕ is unique. For if ϕ' is such that $\phi' \circ \pi = \psi = \phi \circ \pi$, then $\phi'(\overline{v}) = \phi(\overline{v})$ for all $v \in V$. Since π is surjective, $\phi' = \phi$. ■

Let (C, Δ, ε) be a coalgebra, and $J \subseteq C$ be a coideal. We can now prove the following:

Proposition 1.3.6. *There exist linear maps* $\overline{\Delta} : C/J \to C/J \otimes C/J$ *and* $\overline{\varepsilon} : C/J \to k$ *such that* $(C/J, \overline{\Delta}, \overline{\varepsilon})$ *has a unique coalgebra structure.*

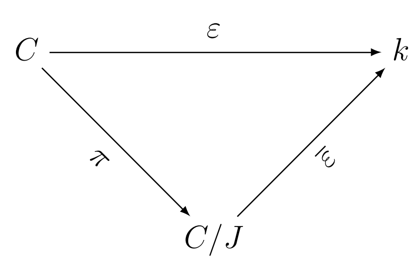
Proof. Consider the linear map $(\pi \otimes \pi)\Delta : C \to C/J \otimes C/J$, and note that $(\pi \otimes \pi)\Delta(J) \subseteq (\pi \otimes \pi)(J \otimes C + C \otimes J) = 0$. Hence $J \subseteq \mathrm{Ker}((\pi \otimes \pi)\Delta)$. So, there exists a unique linear map, $\overline{\Delta} : C/J \to C/J \otimes C/J$ such that $(\pi \otimes \pi)\Delta = \overline{\Delta} \circ \pi$, that is, the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\quad \pi \quad} & C/J \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \overline{\Delta}} \\ C \otimes C & \xrightarrow{\quad \pi \otimes \pi \quad} & C/J \otimes C/J \end{array} \tag{1.7}$$

This map is explicitly given by $\overline{\Delta}(\overline{c}) = \sum \overline{c_1} \otimes \overline{c_2}$ for $c \in C$. Then,

$$\begin{aligned}
&(\overline{\Delta} \otimes I_{C/J})\overline{\Delta}(\overline{c}) \\
=&(\overline{\Delta} \otimes I_{C/J}) \sum \overline{c_1} \otimes \overline{c_2} \\
=&(\overline{\Delta} \otimes I_{C/J}) \sum \pi(c_1) \otimes \pi(c_2) \\
=&\sum \overline{\Delta}(\pi(c_1)) \otimes \pi(c_2) \\
=&\sum (\pi \otimes \pi)\Delta(c_1) \otimes \pi(c_2) \\
=&\sum \pi(c_1) \otimes \pi(c_2) \otimes \pi(c_3), \quad \text{(by the coassociativity of } C\text{)} \\
=&\sum \overline{c_1} \otimes \overline{c_2} \otimes \overline{c_3}.
\end{aligned}$$

Similarly, $(I_{C/J} \otimes \overline{\Delta})\overline{\Delta}(\overline{c}) = \sum \overline{c_1} \otimes \overline{c_2} \otimes \overline{c_3}$. So $\overline{\Delta}$ is coassociative. Furthermore, $J \subseteq \operatorname{Ker}(\varepsilon)$ by definition. Then, there exists a unique linear map $\overline{\varepsilon} : C/J \to k$ such that $\varepsilon = \overline{\varepsilon} \circ \pi$. That is, the following diagram commutes.



(1.8)

Then for any $c \in C$, we have

$$\begin{aligned}
\sum \overline{\varepsilon}(\overline{c_1})\overline{c_2} =& \sum \overline{\varepsilon}(\pi(c_1))\pi(c_2) \\
=&\sum \varepsilon(c_1)\pi(c_2) \\
=&\pi\left(\sum \varepsilon(c_1)c_2\right), \quad \text{(linearity of } \pi\text{)} \\
=&\pi(c), \quad \text{(counit property in } C\text{)} \\
=&\overline{c}.
\end{aligned}$$

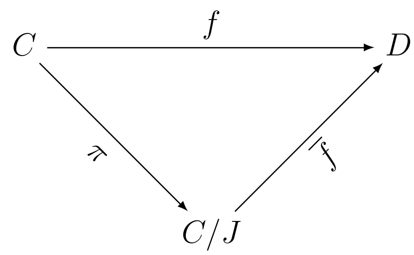
Similarly, $\sum(\overline{c_1})\overline{\varepsilon}(\overline{c_2}) = \overline{c}$, and hence $(C/J, \overline{\Delta}, \overline{\varepsilon})$ is a coalgebra. The uniqueness of the structure follows directly from the uniqueness of the linear maps $\overline{\Delta}$ and $\overline{\varepsilon}$. ■

Corollary 1.3.7. *π is a coalgebra morphism.*

Proof. Follows from the commutativity of diagrams (1.7) and (1.8). ∎

Proposition 1.3.8 (The Fundamental Isomorphism Theorem for Coalgebras)**.** *Let $f : C \to D$ be a surjective morphism of coalgebras and $J = \mathrm{Ker}(f)$. Then $D \cong C/J$.*

Proof. Consider the diagram,



where $\overline{f} : C/J \to D$ via $\overline{c} \mapsto f(c)$. This map is well defined for the same reason as ϕ in the proof of Lemma 1.3.5. Now, suppose $\overline{c} \in \mathrm{Ker}(\overline{f})$. Then $0 = \overline{f}(\overline{c}) = f(c)$. Hence, $c \in \mathrm{Ker}(f) = J$ and $\overline{c} = \overline{0}$, showing $\overline{f}$ is injective. Moreover, since f is surjective, for any $d \in D$, there exists $c \in C$ such that $d = f(c) = \overline{f}(\overline{c})$. Therefore, $\overline{f}$ is a bijection. Finally, using that f is a coalgebra morphism we have,

$$
\begin{aligned}
(\overline{f} \otimes \overline{f})\overline{\Delta}(\overline{c}) &= \sum \overline{f}(\overline{c_1}) \otimes \overline{f}(\overline{c_2}) \\
&= \sum f(c_1) \otimes f(c_2) \\
&= (f \otimes f)\Delta_C(c) \\
&= \Delta_D f(c) \\
&= \Delta_D \overline{f}(\overline{c}),
\end{aligned}
$$

and, $\varepsilon_D \circ \overline{f}(\overline{c}) = \varepsilon_D(f(c)) = \varepsilon_C(c) = \overline{\varepsilon}(\overline{c})$, showing that $\overline{f}$ is a coalgebra morphism. ∎

1.4 Duality

We now emphasize a relationship between algebras and coalgebras, by examining their dual spaces. Explicitly, the dual of a coalgebra is an algebra, and the dual of a finite dimensional algebra is a coalgebra, once defining appropriate, and quite natural, structure maps.

For any k-vector space V we let $V^* = \mathrm{Hom}_k(V, k)$, the linear dual of V. Usually, when there is no ambiguity, we will simply write, $\mathrm{Hom}(V, k)$. Furthermore, if V and W are two k-vector spaces and $\phi : V \to W$ is a morphism of k-vector spaces, then we denote the **transpose** of the map ϕ by $\phi^* : W^* \to V^*$, given by

$$\phi^*(f)(v) = f(\phi(v))$$

for all $f \in W^*$, $v \in V$. Also, we have the following fact.

Proposition 1.4.1. *Let* $\dim(V) = n < \infty$ *and* $\{e_i\}_{i=1}^n$ *be a basis for* V*, then there exists a basis* $\{e_i^*\}_{i=1}^n$ *for* V^* *given by*

$$e_i^*(e_j) = \delta_{ij},$$

where δ_{ij} *is the Kronecker delta. The linearly independent set* $\{e_i^*\}$ *is called a* **dual basis** *to* $\{e_i\}$ *for* V^*. ■

The following Lemma will also be necessary.

Lemma 1.4.2. *Let* V *and* W *be* k*-vector spaces. Define the linear map* $\rho : V^* \otimes W^* \to (V \otimes W)^*$*, given by*

$$\rho(f \otimes g)(v \otimes w) = f(v)g(w),$$

for $f \in V^*, g \in W^*, v \in V, w \in W$*. Then*

(i) ρ *is injective.*

(ii) if V *and* W *are finite dimensional, then* ρ *is an isomorphism.*

Proof. (*i*) Let $x = \sum_i f_i \otimes g_i$, (a finite sum) with the f_i, g_i linearly independent elements of V^*, W^*, respectively, such that $\rho(x) = 0$. Note that this is possible since we can write bases $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$ for V^* and W^*, respectively. Then the family $\{f_\alpha \otimes g_\beta\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$ forms a basis for $V^* \otimes W^*$. We can then choose the finite family $\{f_i \otimes g_i\}_i \subseteq \{f_\alpha \otimes g_\beta\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$. (Note: we may need to take scalar multiples of the f_α to form the f_i, but they are still linearly independent.)

Now suppose there exists $v \in V$ such that $f_i(v) \neq 0$ for some i. Then the sum $\sum_i f_i(v)g_i \neq 0$ by the linear independence of the family, $\{g_i\}_i$. Hence, there must exist $w \in W$ such that

$$\rho(x)(v \otimes w) = \rho\left(\sum_i f_i \otimes g_i\right)(v \otimes w) = \sum_i f_i(v)g_i(w) \neq 0,$$

a contradiction. Thus, $f_i(v) = 0$ for all $v \in V$, and thus $x = \sum_i f_i \otimes g_i = 0$.

(ii). Let $\{v_i\}_{i=1}^n, \{w_j\}_{j=1}^m$ be bases for V and W, respectively. Form the dual bases $\{v_i^*\}_{i=1}^n$ and $\{w_j^*\}_{j=1}^m$ for V^* and W^*, respectively. Then, $\{v_i^* \otimes w_j^*\}_{i,j}$ is a basis for $V^*\otimes W^*$. Thus, $\dim(V^*\otimes W^*) = nm$. On the other hand, the family $\{v_i\otimes w_j\}_{i,j}$ forms a basis for $V\otimes W$, and we can form the dual basis $\{(v_i\otimes w_j)^*\}_{i,j}$ for $(V\otimes W)^*$. Hence $\dim(V \otimes W)^* = nm$ and the two vector spaces must be isomorphic. Since ρ provides an embedding $V^* \otimes W^* \hookrightarrow (V \otimes W)^*$, it must be the desired isomorphism. ■

In order to define a coalgebra structure on the dual of a finite dimensional algebra, we will also need the following corollary.

Corollary 1.4.3. *For any k-vector spaces $V_1, \ldots, V_n$ the map $\theta_n : V_1^* \otimes \cdots \otimes V_n^* \to (V_1 \otimes \cdots \otimes V_n)^*$ given by*

$$\theta_n(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1)\cdots f_n(v_n)$$

is injective. Furthermore, if each V_i is finite dimensional, then θ_n is an isomorphism.

Proof. Follows from induction on the above lemma. ■

Remark 1.4.4. Notice that we have already made use of the map ρ when defining the tensor product of coalgebras. Indeed, for $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ two coalgebras, recall that the counit ε for $C \otimes D$ is given by

$$\varepsilon(c \otimes d) = \rho(\varepsilon_C \otimes \varepsilon_D)(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d),$$

which is exactly the map given in Example 1.1.12. ■

Now, we return to the task at hand. Let (C, Δ, ε) be a coalgebra. We wish to define an algebra structure on C^*. The preceding Lemma ensures that since we can embed $C^* \otimes C^*$ into $(C \otimes C)^*$, we can restrict the map $\Delta^* : (C \otimes C)^* \to C^*$ to a map $m : C^* \otimes C^* \to C^*$ given by

$$m(f \otimes g)(c) = \Delta^* \rho(f \otimes g)(c) = \rho(f \otimes g)\Delta(c) = \sum f(c_1)g(c_2). \tag{1.9}$$

That is, $m = \Delta^* \rho$. Moreover, if we take the map $\varepsilon^* : k^*(\cong k) \to C^*$, we can form the unit map, $U = \varepsilon^* i^{-1}$ where $i : k^* \to k$ is the canonical isomorphism given by $i(f) = f(1)$, (and hence, $i^{-1}(\lambda) = \lambda I$). Then, we can write for any $\lambda \in k$ that $U : k \to C^*$ is given by

$$U(\lambda)(c) = \lambda \varepsilon(c). \tag{1.10}$$

Proposition 1.4.5. *(C^*, m, U) is an algebra.*

Proof. For simplicity, we will denote $m(f \otimes g)$ as $f * g$. Now, for any $f, g, h \in C^*$ and any $c \in C$, making use of the coassociativity in C, we have

$$\begin{aligned}
&((f * g) * h)(c) \\
=&\sum (f * g)(c_1)h(c_2) \\
=&\sum f(c_1)g(c_2)h(c_3) \\
=&\sum f(c_1)(g * h)(c_2) \\
=&(f * (g * h))(c),
\end{aligned}$$

and therefore, m is associative. Now we need to check the unit property. To do this, we verify that $U(1)$ is the identity element of C^*, i.e. that $U(1) * f = f * U(1) = f$ for any $f \in C^*$.

Now, by the counit property of C, we have that $\sum \varepsilon(c_1)c_2 = c$ for all $c \in C$. Then we see,

$$\begin{aligned} &(U(1) * f)(c) \\ =&\sum U(1)(c_1)f(c_2) \\ =&\sum \varepsilon(c_1)f(c_2) \\ =&f\left(\sum \varepsilon(c_1)c_2\right), \quad \text{(by linearity of } f\text{)} \\ =&f(c). \end{aligned}$$

Similarly, using $\sum c_1\varepsilon(c_2) = c$, we obtain $f * U(1) = f$. ■

Now, if we start with an algebra (A, M, u), and try to endow A^* with a coalgebra structure, problems arise. Attempting to proceed as above, we consider the multiplication $M : A \otimes A \to A$, and try to define a comultiplication on A^* via the transpose map $M^* : A^* \to (A \otimes A)^*$. But, since $A^* \otimes A^* \subseteq (A \otimes A)^*$, there is no guarantee that our result will lie in $A^* \otimes A^*$, as desired. If, however, we restrict to *finite dimensional* algebras, then the map $\rho : A^* \otimes A^* \to (A \otimes A)^*$ defined in Lemma 1.4.2 is bijective. So we may define a comultiplication $\delta : A^* \to A^* \otimes A^*$ by $\delta = \rho^{-1}M^*$ and a counit map $E : A^* \to k$ by $E = iu^*$, where i is the canonical isomorphism $k^* \to k$, defined as before.

Facts 1.4.6. *If we write $\delta(f) = \sum_i g_i \otimes h_i$, a finite sum, then*

(i) *$f(ab) = \sum_i g_i(a)h_i(b)$ for all $a, b \in A$.*

(ii) *Moreover, if $\{x_j, y_j\}_j$ is any finite family of elements in A^* such that $\sum_j x_j(a)y_j(b) = f(ab)$, then $\sum_j x_j \otimes y_j = \sum_i g_i \otimes h_i$.*

Proof. (i) First, we note that $M^*(f)(a \otimes b) = f(ab)$ for all $a, b \in A$. Then, the definition of ρ gives that if

$$\delta(f)(a \otimes b) = \rho^{-1} M^*(f)(a \otimes b) = \left(\sum_i g_i \otimes h_i\right)(a \otimes b),$$

then

$$f(ab) = M^*(f)(a \otimes b) = \rho\left(\sum_i g_i \otimes h_i\right)(a \otimes b) = \sum g_i(a) h_i(b)$$

for all $a, b \in A$. Assertion (ii) follows immediately from the injectivity of ρ. ■

Now we are ready to prove the following proposition:

Proposition 1.4.7. *Let (A, M, u) be a finite dimensional algebra. Then (A^*, δ, E) is a coalgebra.*

Proof. Take $f \in A^*$ and write $\delta(f) = \sum_i g_i \otimes h_i$. Also, we will write $\delta(g_i) = \sum_j g'_{i,j} \otimes g''_{i,j}$, and $\delta(h_i) = \sum_j h'_{i,j} \otimes h''_{i,j}$. Then,

$$(\delta \otimes I) \circ \delta(f) = \sum_{i,j} g'_{i,j} \otimes g''_{i,j} \otimes h_i, \text{ and}$$
$$(I \otimes \delta) \circ \delta(f) = \sum_{i,j} g_i \otimes h'_{i,j} \otimes h''_{i,j}.$$

Now using the associativity of A, the fact that $f(ab) = \sum g_i(a) h_i(b)$ for all $a, b \in A$, and the map, $\theta_3 : A^* \otimes A^* \otimes A^* \to (A \otimes A \otimes A)^*$, (as defined in Corollary 1.4.3), we have that,

$$\begin{aligned}
&\theta_3((\delta \otimes I) \circ \delta(f))(a \otimes b \otimes c) \\
=&\theta_3(\sum_{i,j} g'_{i,j} \otimes g''_{i,j} \otimes h_i)(a \otimes b \otimes c) \\
=&\sum_{i,j} g'_{i,j}(a) g''_{i,j}(b) h_i(c) \\
=&\sum_i g_i(ab) h_i(c) \\
=&f(abc)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\theta_3((I\otimes\delta)\circ\delta(f))(a\otimes b\otimes c)\\
=&\theta_3(\sum_{i,j} g_i\otimes h'_{i,j}\otimes h''_{i,j})(a\otimes b\otimes c)\\
=&\sum_{i,j} g_i(a)h'_{i,j}(b)h''_{i,j}(c)\\
=&\sum_{i} g_i(a)h_i(bc)\\
=&f(abc).
\end{aligned}$$

Then, by the injectivity of θ_3, $(\delta\otimes I)\circ\delta(f)=(I\otimes\delta)\circ\delta(f)$, for any $f\in A^*$. Thus, the coassociativity is checked. By the unit property of A, we note that $E(f)=f(1)$ for $f\in A^*$, and hence:

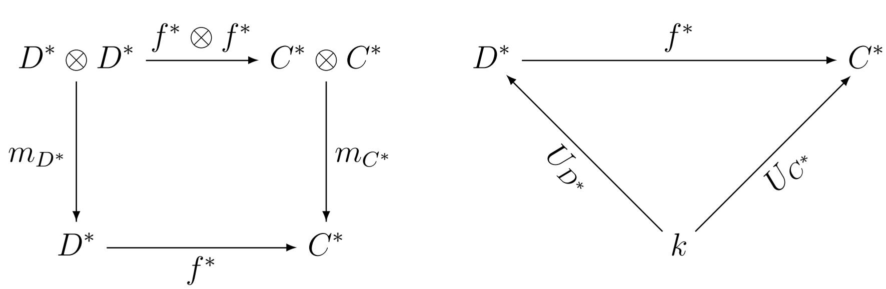
$$\left(\sum_i E(g_i)h_i\right)(a)=\sum_i g_i(1)h_i(a)=f(1\cdot a)=f(a).$$

Similarly, $\sum_i g_iE(h_i)=f$. Thus (A^*,δ,E) is a coalgebra. ■

At this point, it is obvious to ask what happens to algebra morphisms and coalgebra morphisms under dual constructions.

Proposition 1.4.8. *Let* $(C,\Delta_C,\varepsilon_C)$ *and* $(D,\Delta_D,\varepsilon_D)$ *be* k*-coalgebras and* $f:C\to D$ *a coalgebra morphism. Then* (C^*,m_{C^*},U_{C^*}) *and* (D^*,m_{D^*},U_{D^*}) *are algebras (as defined above) and* $f^*:D^*\to C^*$ *is an algebra morphism.*

Proof. We need to show that the following diagrams commute.



Let $x, y \in D^*$ and $c \in C$. Then

$$\begin{aligned}
&(f^* \circ m_{D^*})(x \otimes y)(c) \\
=&\rho(x \otimes y)\Delta_D(f(c)) \\
=&\rho(x \otimes y)\left(\sum f(c)_1 \otimes f(c)_2\right) \\
=&\rho(x \otimes y)\left(\sum f(c_1) \otimes f(c_2)\right) \\
=&\sum (x(f)(c_1))(y(f)(c_2)) \\
=&\rho(x(f) \otimes y(f))\Delta_C(c) \\
=&m_{C^*}(f^*(x) \otimes f^*(y))(c) \\
=&m_{C^*}(f^* \otimes f^*)(x \otimes y)(c),
\end{aligned}$$

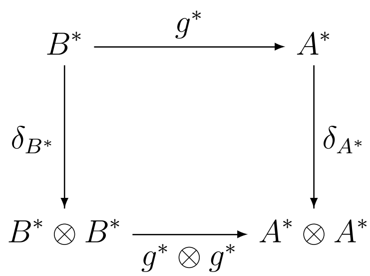
where we have used the fact that f is a coalgebra morphism in the third line. This shows the commutativity of the first diagram above. Now, we use that $\varepsilon_D \circ f = \varepsilon_C$ (f is a coalgebra morphism), and obtain for $\lambda \in k$, $c \in C$,

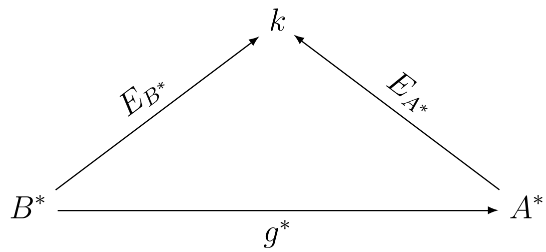
$$\begin{aligned}
(f^* \circ U_D)(\lambda)(c) &= U_D(\lambda)(f(c)) \\
&= \lambda\varepsilon_D(f(c)) \\
&= \lambda\varepsilon_C(c) \\
&= U_C(\lambda)(c).
\end{aligned}$$

Thus, f^* is an algebra morphism. ■

Proposition 1.4.9. *Let (A, M_A, u_A) and (B, M_B, u_B) be finite dimensional k-algebras and $g : A \to B$ an algebra morphism. Then $(A^*, \delta_{A^*}, E_{A^*})$ and $(B^*, \delta_{B^*}, E_{B^*})$ are coalgebras (as defined above) and $g^* : B^* \to A^*$ is a coalgebra morphism.*

Proof. We need to show the following diagrams commute.





Let $x \in B^*$ and $a' \otimes a'' \in A \otimes A$. First, note that if $\delta_{B^*}(x) = \sum_i x_i' \otimes x_i''$, where $x_i', x_i'' \in B^*$, then

$$\begin{aligned} \delta_{B^*}(x)(g \otimes g) = & \sum_i x_i'(g) \otimes x_i''(g) \\ = & \sum_i g^*(x_i') \otimes g^*(x_i'') \\ = & (g^* \otimes g^*) \left(\sum_i x_i' \otimes x_i''\right) \\ = & (g^* \otimes g^*)\delta_{B^*}(x). \end{aligned} \tag{1.11}$$

So,

$$\begin{aligned} & (\delta_{A^*} \circ g^*)(x)(a' \otimes a'') \\ =& \delta_{A^*}(x(g))(a' \otimes a'') \\ =& x(g(a'a'')) \\ =& x(g(a')g(a'')) \\ =& \delta_{B^*}(x)(g(a') \otimes g(a'')) \\ =& \delta_{B^*}(x)(g \otimes g)(a' \otimes a'') \\ =& (g^* \otimes g^*)\delta_{B^*}(x)(a' \otimes a'') \end{aligned}$$

where we have used that g is an algebra morphism in the third line, and (1.11). Hence, the commutativity of the first diagram is checked. Now, for any $x \in B^*$, $b \in B$, we have

$$
\begin{aligned}
(E_{A^*} \circ g^*)(x)(b) &= (E_{A^*} \circ x(g))(b) \\
&= x(g \circ u_A)(b) \\
&= x(u_B)(b) \\
&= E_{B^*}(x)(b),
\end{aligned}
$$

where we have used the property that $g \circ u_A = u_B$ since g is an algebra morphism. This checks the commutativity of the second diagram. ■

We end this subsection with an important result that we will use later, but will not prove. A proof can be found in Dăscălescu, et. al. [4].

Proposition 1.4.10. *Let A be a finite dimensional algebra and C a finite dimensional coalgebra. Then*

(*i*) *A and $(A^*)^*$ are isomorphic as algebras,*

(*ii*) *C and $(C^*)^*$ are isomorphic as coalgebras.* ■

1.5 Group-like and primitive elements

In this section we will define and give some important results for two special classes of elements in a coalgebra. Throughout, let (C, Δ, ε) be a coalgebra.

Definition 1.5.1 (Group-like elements)**.** *Let $g \in C$ such that $g \neq 0$ and $\Delta(g) = g \otimes g$. Then g is called a* **group-like** *element of C. The set of all group-like elements is denoted by $G(C)$.*

Facts 1.5.2. *We have the following facts regarding $G(C)$:*

(*i*) *If $g \in G(C)$, then $\varepsilon(g) = 1$.*

(ii) $G(C)$ is a linearly independent set.

Proof. Assertion (i) follows immediately from the counit property since $g = \sum \varepsilon(g_1)g_2 = \varepsilon(g)g$. To prove (ii), we assume the set $G(C)$ is linearly dependent and seek a contradiction. For, let n be the least natural number so that there exists $g, g_1, \ldots, g_n \in G(C)$, distinct elements such that

$$g = \sum_{i=1}^{n} \alpha_i g_i \tag{1.12}$$

for some scalars $\alpha_i \in k$. Notice that if $n = 1$, then $g = \alpha_1 g_1$ and hence $\varepsilon(g) = \alpha_1 \varepsilon(g_1)$. Then $\alpha_1 = 1$, and by (i) we have $g = g_1$, a contradiction.

So we must have that $n \geq 2$. Moreover, we may assume that all the α_i are non-zero, or else we would obtain equation (1.12) for a natural number smaller than n. Apply Δ to (1.12) to get

$$g \otimes g = \sum_{i=1}^{n} \alpha_i g_i \otimes g_i.$$

Then using (1.12) to replace g on the left hand side we have,

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j g_i \otimes g_j = \sum_{i=1}^{n} \alpha_i g_i \otimes g_i.$$

and therefore

$$0 = \sum_{i \neq j} \alpha_i \alpha_j g_i \otimes g_j + \sum_{i} (\alpha_i^2 - \alpha_i) g_i \otimes g_i.$$

Since $g_1, \ldots, g_n$ are linearly independent (again, otherwise we would have equation (1.12) for a smaller n), we have that $\{g_i \otimes g_j\}_{i,j}$ is linearly independent in $C \otimes C$. Thus, we must have for $i \neq j$, that $\alpha_i \alpha_j = 0$. This is a contradiction since $n \geq 2$ and all $\alpha_i \neq 0$. ■

Let A and B be algebras. Then denote the set of algebra maps from A to B by $\mathrm{Alg}(A, B)$. That is,

$$\mathrm{Alg}(A, B) = \{f \in \mathrm{Hom}(A, B) \mid f(ab) = f(a)f(b),\ \forall a, b \in A\}.$$

Now, if A is a finite dimensional algebra, we can describe the group-like elements of A^* exactly.

Proposition 1.5.3. $G(A^*) = \mathrm{Alg}(A, k)$.

Proof. If $f \in G(A^*)$, then $\delta(f) = f \otimes f$, and by Fact 1.4.6(i), $f(ab) = f(a)f(b)$ for all $a, b \in A$. Moreover, $f(1) = E(f) = 1$, so $f \in \mathrm{Alg}(A, k)$. The reverse inclusion is similar. ■

Definition 1.5.4 (Primitive elements)**.** *Let (C, Δ, ε) be a coalgebra and $c \neq 0$ in C such that $\Delta(c) = 1 \otimes c + c \otimes 1$. Then c is called a* **primitive** *element of C. The set of all primitive elements is denoted by $P(C)$. We can similarly define, for $x, y \in C$ the set of x, y-***primitive** *elements, denoted by $P_{x,y}(C)$, where if $c \in P_{x,y}(C)$ then $\Delta(c) = x \otimes c + c \otimes y$.*

Facts 1.5.5. *We have the following facts about primitive elements:*

(i) *Let $x \in G(C)$, $0 \neq y \in C$. If $c \in P_{x,y}(C)$, then $\varepsilon(c) = 0$. (Similarly, if $y \in G(C)$, $0 \neq x \in C$.)*

(ii) *For any $x \in G(C)$ and $0 \neq y \in C$, $P_{x,y}(C)$ is a coideal of C.*

Proof. (i) The counit property gives that $c = \sum \varepsilon(c_1)c_2$. So $c = \varepsilon(x)c + \varepsilon(c)y = c + \varepsilon(c)y$. Thus $\varepsilon(c) = 0$. To prove (ii), we note that $P_{x,y}(C)$ is a subspace of C, since for any $c, d \in P_{x,y}(C)$,

$$\begin{aligned}\Delta(c+d) &= \Delta(c) + \Delta(d) \\ &= x \otimes c + c \otimes y + x \otimes d + d \otimes y \\ &= x \otimes (c+d) + (c+d) \otimes y.\end{aligned}$$

Then, for any $c \in P_{x,y}(C)$, $\Delta(c) = x \otimes c + c \otimes y \in C \otimes P_{x,y}(C) + P_{x,y}(C) \otimes C$ and part (i) gives $P_{x,y}(C) \subseteq \mathrm{Ker}(\varepsilon)$. ■

Chapter 2: Bialgebras and Hopf algebras

We will now examine k-vector spaces that are simultaneously endowed with both an algebra and coalgebra structure. In particular, we will want to investigate spaces in which the two structures relate in a "nice" way. Let (H, M, u) be an algebra which is also a coalgebra, (H, Δ, ε). Recall that we have a natural algebra structure induced on $H \otimes H$, as well as a coalgebra structure described in Proposition 1.2.4. We must also recall the canonical coalgebra structure on the field k as described in Example 1.1.8.

2.1 Bialgebras

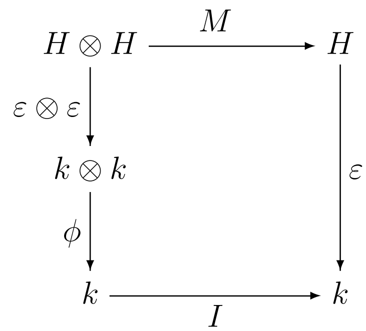
Proposition 2.1.1. *Let (H, M, u) be an algebra which is simultaneously a coalgebra, (H, Δ, ε). Then the following are equivalent:*

(i) *M and u are coalgebra morphisms.*

(ii) *Δ and ε are algebra morphisms.*

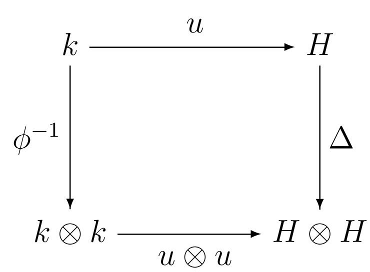
Proof. $(i) \Rightarrow (ii)$ M is a coalgebra morphism exactly when the following diagrams commute:

$$\begin{array}{ccc}
H \otimes H & \xrightarrow{\quad M \quad} & H \\
\Delta \otimes \Delta \downarrow & & \\
H \otimes H \otimes H \otimes H & & \downarrow \Delta \\
I \otimes T \otimes I \downarrow & & \\
H \otimes H \otimes H \otimes H & \xrightarrow[M \otimes M]{} & H \otimes H
\end{array} \tag{2.1}$$

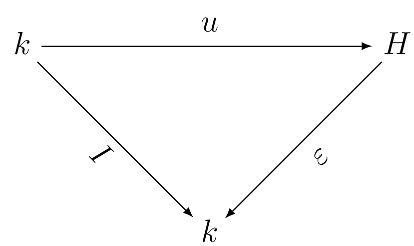


(2.2)

where $\phi : k \otimes k \to k$ is the canonical isomorphism $\phi(\alpha \otimes \beta) = \alpha\beta$. Moreover, u is a coalgebra morphism exactly when the following diagrams commute:



(2.3)



(2.4)

Where ϕ^{-1} is the comultiplication map for k defined in Example 1.1.8, and given by $\alpha \mapsto \alpha \otimes 1$, for $\alpha \in k$. We note that Δ is an algebra morphism if and only if diagrams (2.1) and (2.3) commute and ε is an algebra morphism if and only if diagrams (2.2) and (2.4) commute. Then, using the same diagrams, the reverse implication is also clear. ■

Remark 2.1.2. For easier calculation in the future, we note that in the sigma notation, if Δ and ε are to be algebra morphisms, we must have

$$\Delta(xy) = \sum x_1 y_1 \otimes x_2 y_2, \quad \Delta(1) = 1 \otimes 1,$$
$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y), \qquad \varepsilon(1) = 1,$$

for all $x, y \in H$. ■

We are now in a position to define a bialgebra.

Definition 2.1.3. *Let H be a k-vector space with algebra structure (H, M, u) and coalgebra structure (H, Δ, ε) such that M and u are coalgebra morphisms (and hence Δ, ε are algebra morphisms). Then $(H, M, u, \Delta, \varepsilon)$ is called a* **bialgebra**.

We return to our prototypical example.

Example 2.1.4. Let G be a group and kG the associated group algebra. Let kG have a coalgebra structure as in Example 1.1.6. Then kG is a bialgebra. To verify this, we check the conditions of Remark 2.1.2. Let $g, h \in G$. Then, $gh \in G$, so

$$\begin{aligned} \Delta(gh) &= gh \otimes gh \\ &= (g \otimes g)(h \otimes h) \\ &= \Delta(g)\Delta(h), \end{aligned}$$

and $\varepsilon(g)\varepsilon(h) = 1 \cdot 1 = 1 = \varepsilon(gh)$. Furthermore, $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$, by definition. ■

Example 2.1.5. Let $(B, M, u, \Delta, \varepsilon)$ be a bialgebra. Then we can form bialgebras B^{op} and B^{cop}, where B^{op} (resp. B^{cop}) is the vector space B, but with algebra (resp. coalgebra) structure given by the opposite algebra (resp. coopposite coalgebra) defined in Example 1.1.10 (resp. 1.1.11). B^{copop} has both the opposite and coopposite structures. It is not difficult to check that each of these are bialgebras. ■

We have shown that the structures of (finite dimensional) algebras and coalgebras are, in fact, dual constructions. This leads to another method of obtaining new bialgebras from existing ones.

Proposition 2.1.6. *Let* $(H, M, u, \Delta, \varepsilon)$ *be a finite dimensional bialgebra. Then* H^* *is a bialgebra with algebra structure dual to the coalgebra structure on* H (*as defined by Proposition 1.4.5*), *and coalgebra structure dual to the algebra structure on* H (*as defined by Proposition 1.4.7*).

Proof. We have already shown that (H^*, m, U) is an algebra when we define the multiplication $m = \Delta^*\rho$ and unit $U = \varepsilon^* i^{-1}$. We have also shown that (H^*, δ, E) is a coalgebra when we define the comultiplication $\delta = \rho^{-1}M^*$ and counit $E = iu^*$.

We need only to show that m and U are coalgebra morphisms (or equivalently, δ and E are algebra morphisms), but this follows readily from Proposition 1.4.9. Indeed, since H is a bialgebra, Δ and ε are algebra morphisms, and thus Δ^* and ε^* are coalgebra morphisms. Since m can be thought of simply as a restriction of Δ^*, m is also a coalgebra morphism. Similarly, U is a coalgebra morphism since it is ε^* following the canonical isomorphism i^{-1}. ■

For the remainder of the section, let H and B be bialgebras, with respective structure maps $(M_H, u_H, \Delta_H, \varepsilon_H)$ and $(M_B, u_B, \Delta_B, \varepsilon_B)$.

Definition 2.1.7. *The k-linear map* $f : H \to B$ *is called a* **bialgebra morphism** *if* f *is both an algebra morphism and coalgebra morphism.*

Definition 2.1.8. *A subspace* $J \subseteq H$ *is called*

(*i*) *a* **bi-subalgebra** *if it is a subalgebra and co-subalgebra.*

(*ii*) *a* **biideal** *if it is an ideal and coideal.*

The following propositions are immediate consequences of the preceding definitions:

Proposition 2.1.9. *If* H *and* B *are finite dimensional, and* $f : H \to B$ *is a bialgebra morphism, then* $f^* : B^* \to H^*$ *is a bialgebra morphism.*

Proof. Since H and B are finite dimensional, we can be sure that H^* and B^* are indeed bialgebras. Then, f^* must be an algebra morphism since f is a coalgebra morphism. Similarly, f^* must also be a coalgebra morphism since f is an algebra morphism. ■

Proposition 2.1.10. *Let $J \subseteq H$ be a biideal. Then $(H/J, \overline{M}, \overline{u}, \overline{\Delta}, \overline{\varepsilon})$ is a bialgebra.*

Proof. Since J is an ideal, it defines the unique algebra structure $(H/J, \overline{M}, \overline{u})$, and similarly, since J is a coideal we obtain the unique coalgebra structure $(H/J, \overline{\Delta}, \overline{\varepsilon})$.

Let $\pi : H \to H/J$ be the canonical vector space projection and write $\pi(h) = \overline{h}$ as the coset of h modulo J. Recall that π is both an algebra and coalgebra morphism. Now we verify the conditions of Remark 2.1.2,

$$\begin{aligned}
&\overline{\Delta}((\overline{x})(\overline{y}))\\
=&\overline{\Delta}(\pi(x)\pi(y))\\
=&\overline{\Delta}(\pi(xy)), \quad (\pi \text{ is an algebra morphism})\\
=&(\pi \otimes \pi)\Delta(xy), \quad (\pi \text{ is a coalgebra morphism})\\
=&(\pi \otimes \pi)\left(\sum x_1y_1 \otimes x_2y_2\right), \quad (\Delta \text{ is an algebra morphism})\\
=&\sum \pi(x_1y_1) \otimes \pi(x_2y_2)\\
=&\sum \pi(x_1)\pi(y_1) \otimes \pi(x_2)\pi(y_2), \quad (\pi \text{ is an algebra morphism})\\
=&\sum (\overline{x_1})(\overline{y_1}) \otimes (\overline{x_2})(\overline{y_2}).
\end{aligned}$$

Also,

$$
\begin{aligned}
&\overline{\varepsilon}((\overline{x})(\overline{y}))\\
=&\overline{\varepsilon}(\pi(x)\pi(y))\\
=&\overline{\varepsilon}(\pi(xy)), \quad (\pi \text{ is an algebra morphism})\\
=&\varepsilon(xy), \quad (\pi \text{ is a coalgebra morphism})\\
=&\varepsilon(x)\varepsilon(y), \quad (\varepsilon \text{ is an algebra morphism})\\
=&\overline{\varepsilon}(\pi(x))\overline{\varepsilon}(\pi(y)), \quad (\pi \text{ is a coalgebra morphism})\\
=&\overline{\varepsilon}(\overline{x})\overline{\varepsilon}(\overline{y}).
\end{aligned}
$$

It is easy to check that $\overline{\Delta}(\overline{1}) = \overline{1} \otimes \overline{1}$ and $\overline{\varepsilon}(\overline{1}) = 1$. ■

Proposition 2.1.11. *Let $f : H \to B$ be a bialgebra morphism. Then,*

(*i*) *$f(H)$ is a bi-subalgebra of B.*

(*ii*) $\mathrm{Ker}(f)$ *is a biideal of H.*

Proof. Combine the analogous results for algebras and coalgebras. ■

Proposition 2.1.12 (The Fundamental Isomorphism Theorem for Bialgebras)**.** *Let $f : H \to B$ be a surjective bialgebra morphism and $J = \mathrm{Ker}(f)$. Then $B \cong H/J$.*

Proof. Combine the analogous results for algebras and coalgebras, plus Proposition 2.1.10. ■

2.2 Convolution, antipodes, and Hopf algebras

We begin by letting (C, Δ, ε) be a coalgebra and (A, M, u) be an algebra. Now, consider $\mathrm{Hom}(C, A)$, the family of k-linear maps from C to A. We can define an algebra structure on $\mathrm{Hom}(C, A)$ via the so-called **convolution product**. We will denote the product of $f, g \in \mathrm{Hom}(C, A)$ by $f * g$. The product is explicitly given by

$$(f * g)(c) = M(f \otimes g)\Delta(c) = \sum f(c_1)g(c_2).$$

Note that it was necessary to have C a coalgebra and A an algebra in order to define the convolution product. Now, we should justify the title "algebra" for $\mathrm{Hom}(C, A)$ under convolution by checking that the product is associative. Indeed, for $f, g, h \in \mathrm{Hom}(C, A)$ and $c \in C$, we have

$$\begin{aligned}((f * g) * h)(c) &= \sum (f * g)(c_1)h(c_2) \\ &= \sum f(c_1)g(c_2)h(c_3) \\ &= \sum f(c_1)(g * h)(c_2) \\ &= (f * (g * h))(c),\end{aligned}$$

where we have made use of the coassociativity of C. Additionally, we note that $u\varepsilon$ is the identity element of $\mathrm{Hom}(C, A)$, since for any $f \in \mathrm{Hom}(C, A), c \in C$,

$$\begin{aligned}(f * u\varepsilon)(c) &= \sum f(c_1)u(\varepsilon(c_2)) \\ &= \sum f(c_1)\varepsilon(c_2)u(1) \\ &= \sum f(c_1)\varepsilon(c_2) \\ &= f\left(\sum c_1\varepsilon(c_2)\right) \\ &= f(c),\end{aligned}$$

where we have used that u is linear in the second line, and the counit property in the last line. Similarly, using $\sum \varepsilon(c_1)c_2 = c$ we obtain $u\varepsilon * f = f$. We have already considered a special case of the convolution product in defining an algebra structure on C^*, that is, $\mathrm{Hom}(C, k)$. Now, consider the special case in which $(H, M, u, \Delta, \varepsilon)$ is a bialgebra. Then it makes sense to talk about the algebra $\mathrm{Hom}(H^c, H^a)$, where we think of H^c as the underlying coalgebra and H^a as the underlying algebra.

Definition 2.2.1. *Let H be a bialgebra. A map $S \in \mathrm{Hom}(H^c, H^a)$ is called an* **antipode** *if it is the inverse to the identity map under convolution. H, together with an antipode, is called a* **Hopf algebra**.

We have the following facts.

Facts 2.2.2. *Let $(H, M, u, \Delta, \varepsilon, S)$ be a Hopf algebra and $h, g \in H$. Then,*

(*i*) (Antipode Property) $\sum S(h_1)h_2 = \sum h_1 S(h_2) = \varepsilon(h)1$,

(*ii*) $S(hg) = S(g)S(h)$ *and* $S(1) = 1$. *That is, S is an anti-algebra morphism,*

(*iii*) $\Delta(S(h)) = \sum S(h)_1 \otimes S(h)_2 = \sum S(h_2) \otimes S(h_1)$ *and* $\varepsilon(S(h)) = \varepsilon(h)$. *That is, S is an anti-coalgebra morphism,*

Proof. (*i*) This follows directly from the definition of S. For, since $S * I = I * S = u\varepsilon$, then for any $h \in H$,

$$\begin{aligned}(S * I)(h) &= \sum S(h_1)h_2 \\ (I * S)(h) &= \sum h_1 S(h_2) \\ u(\varepsilon(h)) &= \varepsilon(h)u(1) = \varepsilon(h)1.\end{aligned}$$

(*ii*) Let $H \otimes H$ have the tensor product of coalgebras structure and H an algebra structure. Then, it makes sense to form the algebra $\mathrm{Hom}(H \otimes H, H)$. Define the maps $X, Y \in \mathrm{Hom}(H \otimes H, H)$ by $X(h \otimes g) = S(hg)$, $Y(h \otimes g) = S(g)S(h)$. Let $h, g \in H$, and consider,

$$\begin{aligned}
&(X * M)(h \otimes g)\\
=&\sum X((h \otimes g)_1)M((h \otimes g)_2)\\
=&\sum X(h_1 \otimes g_1)M(h_2 \otimes g_2), \quad \text{(comultiplication in } H \otimes H)\\
=&\sum S(h_1 g_1)h_2 g_2\\
=&\sum S((hg)_1)(hg)_2, \quad (M \text{ is a coalgebra morphism})\\
=&\varepsilon_H(hg)1, \quad \text{(antipode property)}\\
=&\varepsilon_H(h)\varepsilon_H(g)1\\
=&(u_H \varepsilon_{H\otimes H})(h \otimes g).
\end{aligned}$$

So X is a left inverse of M. Now, we consider

$$\begin{aligned}
&(M * Y)(h \otimes g)\\
=&\sum M((h \otimes g)_1)Y((h \otimes g)_2)\\
=&\sum M(h_1 \otimes g_1)Y(h_2 \otimes g_2), \quad \text{(comultiplication in } H \otimes H)\\
=&\sum h_1 g_1 S(g_2)S(h_2)\\
=&\sum h_1(\varepsilon_H(g)1)S(h_2), \quad \text{(antipode property)}\\
=&\sum h_1 S(h_2)\varepsilon_H(g)1\\
=&\varepsilon_H(h)\varepsilon_H(g)1, \quad \text{(antipode property)}\\
=&(u_H \varepsilon_{H\otimes H})(h \otimes g)
\end{aligned}$$

So Y is a right inverse for M. Since $\text{Hom}(H \otimes H, H)$ forms an algebra, and therefore has unique inverses, $X = Y$, and we conclude that $S(hg) = S(g)S(h)$. Consider the element $1 \in H$. Applying (i), we get that $S(1)1 = \varepsilon(1)1$, that is, $S(1) = 1$.

(iii) Similar to (ii), we think of H with a coalgebra structure and $H \otimes H$ with an algebra structure to define $\text{Hom}(H, H \otimes H)$. Let $F, G \in \text{Hom}(H, H \otimes H)$ with

$F(h) = \sum S(h)_1 \otimes S(h)_2 = \Delta(S(h))$ and $G(h) = \sum S(h_2)S(h_1)$. Then for any $h \in H$,

$$\begin{aligned}
&(\Delta * F)(h)\\
=&\sum \Delta(h_1)F(h_2)\\
=&\sum \Delta(h_1)\Delta(S(h_2))\\
=&\sum \Delta(h_1 S(h_2)), \quad (\Delta \text{ is an algebra morphism})\\
=&\Delta\left(\sum h_1 S(h_2)\right), \quad (\text{linearity of } \Delta)\\
=&\Delta(\varepsilon(h)1), \quad (\text{antipode property})\\
=&\varepsilon(h)1 \otimes 1\\
=&(u_{H\otimes H}\varepsilon_H)(h).
\end{aligned}$$

Thus F is a right inverse for Δ. Moreover,

$$\begin{aligned}
&(G * \Delta)(h)\\
=&\sum G(h_1)\Delta(h_2)\\
=&\sum (S((h_1)_2) \otimes S((h_1)_1))((h_2)_1 \otimes (h_2)_2)\\
=&\sum (S(h_2) \otimes S(h_1))(h_3 \otimes h_4), \quad (\text{generalized coassociativity})\\
=&\sum S(h_2)h_3 \otimes S(h_1)h_4\\
=&\sum S((h_2)_1)(h_2)_2 \otimes S(h_1)h_3, \quad (\text{generalized coassociativity})\\
=&\sum \varepsilon(h_2)1 \otimes S(h_1)h_3, \quad (\text{antipode property})\\
=&\sum 1 \otimes S(h_1)\varepsilon((h_2)_1)(h_2)_2, \quad (\text{generalized coassociativity})\\
=&\sum 1 \otimes S(h_1)h_2, \quad (\text{counit property})\\
=&1 \otimes \varepsilon(h)1, \quad (\text{antipode property})\\
=&\varepsilon(h)1 \otimes 1\\
=&(u_{H\otimes H}\varepsilon_H)(h).
\end{aligned}$$

Hence G is a left inverse for Δ, and $G = F$. Finally, for any $h \in H$ we have that

$\varepsilon(h)1 = \sum h_1 S(h_2)$. Now, applying ε to both sides we obtain

$$\begin{aligned}\varepsilon(h) &= \sum \varepsilon(h_1)\varepsilon(S(h_2)), \quad (\varepsilon \text{ is an algebra morphism})\\ &= \sum \varepsilon(\varepsilon(h_1)S(h_2)), \quad (\text{linearity of counit})\\ &= \sum \varepsilon(S(\varepsilon(h_1)h_2)), \quad (\text{linearity of antipode})\\ &= \varepsilon\left(S\left(\sum \varepsilon(h_1)h_2\right)\right), \quad (\text{again using linearity})\\ &= \varepsilon(S(h)),\end{aligned}$$

by the counit property. ■

We now define the ideals in Hopf algebras.

Definition 2.2.3. *Let H be a Hopf algebra. A subspace $J \subseteq H$ is called*

(i) a **Hopf-subalgebra** *if it is a bi-subalgebra and $S(J) \subseteq J$.*

(ii) a **Hopf ideal** *if it is a biideal and $S(J) \subseteq J$.*

It seems reasonable for ideals and maps to have some preserving effect on antipodes. With this in mind, we turn our attention to maps between Hopf algebras. Let H and B be Hopf algebras with antipodes S_H and S_B, respectively. We will say that a linear map $f : H \to B$ **preserves antipodes** whenever $S_B f = f S_H$. Let f be a bialgebra morphism. We can define the algebra $\mathrm{Hom}(H, B)$ with identity element $u_B\varepsilon_H$. Of course, $f \in \mathrm{Hom}(H, B)$, so consider

$$\begin{aligned}&((S_B f) * f)(h)\\ =&\sum (S_B f)(h_1) f(h_2)\\ =&\sum S_B(f(h)_1) f(h)_2, \quad (f \text{ is a coalgebra morphism})\\ =&\varepsilon_B(f(h))1, \quad (\text{antipode property for } B)\\ =&\varepsilon_H(h)1, \quad (f \text{ is a coalgebra morphism})\\ =&(u_B\varepsilon_H)(h),\end{aligned}$$

and

$$
\begin{aligned}
&(f * (fS_H))(h) \\
=&\sum f(h_1)(fS_H(h_2)) \\
=&\sum f(h_1 S_H(h_2)), \quad (f \text{ is an algebra morphism}) \\
=&f(\varepsilon_H(h)1), \quad (\text{antipode property for } H) \\
=&\varepsilon_H(h)1, \quad (f \text{ is an algebra morphism}) \\
=&(u_B \varepsilon_H)(h),
\end{aligned}
$$

showing that $S_B f$ is a left inverse of f and fS_H is a right inverse for f. Thus $S_B f = fS_H$, and we have proven:

Proposition 2.2.4. *If H and B are Hopf algebras and $f : H \to B$ is a bialgebra morphism, then f preserves antipodes.* ∎

Therefore, the following definition is justified:

Definition 2.2.5. *Let H and B be Hopf algebras and $f : H \to B$. f is called a* **Hopf morphism** *if f is a bialgebra morphism.*

The following proposition is immediate.

Proposition 2.2.6. *If $f : H \to B$ is a Hopf morphism, then*

(*i*) *$f(H)$ is a Hopf-subalgebra of B.*

(*ii*) Ker(f) *is a Hopf ideal of H.*

Proof. (*i*) f is a bialgebra morphism, so we have that $f(H)$ is a bi-subalgebra. Moreover, f preserves antipodes, so $S_B(f(H)) = f(S_H(H)) \subseteq f(H)$.

(*ii*) We need only to show that $S(\mathrm{Ker}(f)) \subseteq \mathrm{Ker}(f)$. Indeed, since f is a Hopf morphism, it preserves antipodes. Then for $h \in \mathrm{Ker}(f)$, $f(S_H(h)) = S_B(f(h)) = S_B(0) = 0$. Hence, $S_H(h) \in \mathrm{Ker}(f)$. ∎

Also, we have the analogous results for quotient structures.

Proposition 2.2.7. *Let $(H, M, u, \Delta, \varepsilon, S)$ be a Hopf algebra and J a Hopf ideal. Then there is a unique Hopf algebra structure on H/J.*

Proof. We already know from Proposition 2.1.10 that H/J has a unique bialgebra structure. It becomes a Hopf algebra with induced map $\overline{S} : H/J \to H/J$ given by $\overline{S}(\overline{h}) = \overline{S(h)}$. This map makes sense because J is a Hopf ideal, that is, $S(J) \subseteq J$. Then we can check, for $h \in H$,

$$\begin{aligned}\sum \overline{S}(\overline{h_1})\overline{h_2} &= \sum \overline{S(h_1)}(\overline{h_2}) \\ &= \sum \pi(S(h_1))\pi(h_2) \\ &= \pi\left(\sum S(h_1)h_2\right) \\ &= \pi(\varepsilon(h)1), \quad \text{(antipode property of } H) \\ &= \varepsilon(h)\overline{1} \\ &= \overline{\varepsilon}(\overline{h})\overline{1}.\end{aligned}$$

Similarly, $\sum(\overline{h_1})\overline{S}(\overline{h_2}) = \overline{\varepsilon}(\overline{h})\overline{1}$. ■

Proposition 2.2.8 (The Fundamental Isomorphism Theorem for Hopf Algebras)**.** *Let H, B be Hopf algebras with antipodes S_H, S_B, respectively. Let $f : H \to B$ be a surjective Hopf morphism and $J = \mathrm{Ker}(f)$. Then $B \cong H/J$.*

Proof. Use the map $\overline{f}$ from the proof of Proposition 1.3.8. We know that $\overline{f}$ is a bialgebra morphism and therefore a Hopf morphism. ■

Just as with bialgebras, we may create new Hopf algebras from existing ones, using the dual space.

Proposition 2.2.9. *If H is a finite dimensional Hopf algebra with antipode S, then H^* is a Hopf algebra with antipode S^*.*

Proof. It remains only to show that S^* is an antipode. As usual, we will denote the multiplication and unit in H^* by m and U, respectively, and the comultiplication and counit by δ and E, respectively. We write for any $h^* \in H^*$, that $\delta(h^*) = \sum h_1^* \otimes h_2^*$ by the sigma notation. Also, recall from Fact 1.4.6(i) that for any $x, y \in H$, $h^*(xy) = \sum h_1^*(x)h_2^*(y)$. Also, $E(h^*) = h^*(1)$, so we have for any $h \in H$,

$$\begin{aligned}
\left(\sum S^*(h_1^*)h_2^*\right)(h) &= \sum S^*(h_1^*)(h_1)h_2^*(h_2) \\
&= \sum h_1^*(S(h_1))h_2^*(h_2) \\
&= \sum h^*(S(h_1)h_2) \\
&= h^*(\varepsilon(h)1), \quad \text{(antipode property of } H) \\
&= \varepsilon(h)h^*(1) \\
&= E(1)\varepsilon(h)
\end{aligned}$$

Similarly, $\sum h_1^* S^*(h_2^*) = E(1)\varepsilon$, so S is an antipode of H^*. ■

Remark 2.2.10. Let H be a Hopf algebra. We will call H commutative if its underlying algebra is commutative. We will call H cocommutative if its underlying coalgebra is cocommutative. Additionally, we can talk about Hopf algebras with neither or both properties. ■

We have the following proposition.

Proposition 2.2.11. *Let H be a finite dimensional Hopf algebra. Then,*

(i) *H is commutative if and only if H^* is cocommutative.*

(ii) *H is cocommutative if and only if H^* is commutative.*

Proof. We will show the forward implications for both (i) and (ii). The reverse implications then follow from Proposition 1.4.10. For (i), let $f \in H^*$, and $x, y \in H$. Using Fact 1.4.6(i), we see $\sum f_1(x)f_2(y) = f(xy) = f(yx)\sum f_2(x)f_1(y)$. Hence

$\sum f_1 \otimes f_2 = \sum f_2 \otimes f_1$, that is, H^* is cocommutative. For (ii), let $f, g \in H^*$ and $x \in H$. Then, $m(f \otimes g)(x) = \sum f(x_1)g(x_2) = \sum f(x_2)g(x_1) = \sum g(x_1)f(x_2) = m(g \otimes f)(x)$. Hence, H^* is commutative. ■

2.3 Some examples of Hopf algebras

In this section we give various examples and small results for Hopf algebras, beginning with our prototype.

Example 2.3.1 (The Group Algebra)**.** Let G be a multiplicative group. Then kG becomes a Hopf algebra with bialgebra structure from Example 2.1.4 and antipode $S : kG \to kG$ given by $S(g) = g^{-1}$ for all $g \in G$. S is indeed an antipode. For consider $g \in G$,

$$\sum S(g_1)g_2 = S(g)g = g^{-1}g = 1_{kG} = \varepsilon(g)1_{kG}.$$

Similarly, $\sum g_1 S(g_2) = \varepsilon(g)1_{kG}$. ■

Example 2.3.2 (The coopposite Hopf algebra)**.** As in 2.1.5, if H is a Hopf algebra with antipode S, then certainly H^{cop} is a bialgebra. Suppose that H^{cop} is a Hopf algebra with antipode $\overline{S}$. This means that $\sum(\overline{S}h_2)h_1 = \sum h_2(\overline{S}h_1) = \varepsilon(h)1$ for all $h \in H$, and since $\overline{S}$ is an antipode for H^{cop} it is an anti-algebra and anti-coalgebra morphism. $\overline{S}$ is called a **twisted antipode** for H. Moreover, for any $h \in H$ we have

$$\begin{aligned} \overline{S}Sh &= \sum \overline{S}S(\varepsilon(h_2)h_1) &= \sum \varepsilon(h_2)\overline{S}Sh_1 &= \sum h_3(\overline{S}h_2)\overline{S}Sh_1 \\ &= \sum h_3\overline{S}((Sh_1)h_2) &= \sum h_2\overline{S}(\varepsilon(h_1)) &= \sum \varepsilon(h_1)h_2 = h. \end{aligned}$$

Similarly, $S\overline{S}h = h$ for any $h \in H$. Therefore, when this situation occurs, we must have that S and $\overline{S}$ are composition inverses. This gives a necessary condition for H^{cop} to be a Hopf algebra. To complete the example, we have the following proposition. ■

Proposition 2.3.3. *Let H be a bialgebra. The following are equivalent:*

(i) H is a Hopf algebra with composition invertible antipode S.

(ii) H^{cop} is a Hopf algebra with composition invertible antipode $\overline{S}$.

That is, H^{cop} is a Hopf algebra if and only if H has a composition invertible antipode.

Proof. We show that $(i) \Rightarrow (ii)$; the reverse direction is similar. Let S' be the composition inverse of S. It is not difficult to check that S' must be an anti-algebra and anti-coalgebra morphism. Now take any $h \in H$ and consider

$$\sum (S'h_2)h_1 = \sum (S'h_2)(S'Sh_1) = S'\left(\sum (Sh_1)h_2\right) = S'(\varepsilon(h)1) = \varepsilon(h)1.$$

Similarly, $\sum h_2(S'h_1) = \varepsilon(h)1$, so S' is an antipode for H^{cop}. By the analysis of the previous example, $S' = \overline{S}$. ■

The following example is due to Sweedler and is the smallest non-commutative, non-cocommutative Hopf algebra.

Example 2.3.4 (The Sweedler algebra)**.** Assume that $\mathrm{ch}(k) \neq 2$. Let H be the algebra given by the two generators g and x with the following relations:

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx$$

Then H has dimension four as a vector space, with basis $\{1, g, x, gx\}$. Define a coalgebra structure by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = g \otimes x + x \otimes 1.$$

That is, $g \in G(H)$ and $x \in P_{g,1}(H)$ so by Facts 1.5.2(i) and 1.5.5(i) we must have $\varepsilon(g) = 1$ and $\varepsilon(x) = 0$. Finally, define $S : H \to H$ to be the antipode given by

$$S(g) = g, \quad S(x) = -gx.$$

One can check that these relations define a Hopf algebra. ■

The Sweedler algebra was later generalized by Taft [22] to any dimension that is a perfect square.

Example 2.3.5 (The Taft algebras)**.** Let $n \geq 2$ and λ a primitive n-th root of unity in k (in particular this means that $\mathrm{ch}(k)$ does not divide n). Again, let $H_n(\lambda)$ be the algebra generated by g and x, with relations

$$g^n = 1, \quad x^n = 0, \quad xg = \lambda gx.$$

Then $H_n(\lambda)$ is an algebra with vector space basis $\{g^i x^j\}_{i,j=0}^{n-1}$. Thus $\dim(H_n(\lambda)) = n^2$. We introduce a coalgebra structure just as in Example 2.3.4:

$$\Delta(g) = g \otimes g, \quad \Delta(x) = g \otimes x + x \otimes 1,$$

so that again we must have, $\varepsilon(g) = 1$ and $\varepsilon(x) = 0$. The antipode is given by $S(g) = g^{n-1}$ and $S(x) = -g^{n-1}x$. Notice that when $n = 2$ we obtain the Sweedler algebra. ■

Remark 2.3.6. (Antipodes of Arbitrary Order.) More than the fact that $H_n(\lambda)$ is a generalization of the Sweedler algebra, it has historical importance. Taft constructed the algebras $H_n(\lambda)$ to give explicit examples of Hopf algebras having antipodes of arbitrarily large orders. We say that the antipode has **order** m, whenever m is the least positive integer such that $S^m = I$, under composition. In particular, one can check that the order of the antipode in $H_n(\lambda)$ is $2n$. To see this, we first check the effect of composition of the antipode on the group-like generator g. We can show by induction on m that

$$S^m(g) = (\underbrace{S \circ \cdots \circ S}_{m})(g) = g^{(n-1)^m}.$$

For $m = 1$, $S(g) = g^{n-1}$ by definition. Then if the assertion is true for all natural numbers $< m$, since S is an anti-algebra morphism,

$$S^m(g) = S(S^{m-1}(g)) = S(g^{(n-1)^{m-1}}) = (S(g))^{(n-1)^{m-1}} = (g^{n-1})^{(n-1)^{m-1}} = g^{(n-1)^m}.$$

Now, since

$$(n-1)^m = \sum_{k=0}^{m} \binom{m}{k} n^{m-k}(-1)^k,$$

we see that $(n-1)^m \equiv (-1)^m \mod n$. Therefore,

$$S^m(g) = \begin{cases} g & , \quad \text{if } m \text{ even} \\ g^{n-1} & , \quad \text{if } m \text{ odd} \end{cases}.$$

So if m is to be the order of the antipode, then m must be even. We also have the following recursive relation for $S^m(x)$:

$$S^m(x) = \begin{cases} -S^{m-1}(x)g & , \quad \text{if } m \text{ even} \\ \\ -g^{n-1}S^{m-1}(x) & , \quad \text{if } m \text{ odd} \end{cases}.$$

To check this, we note that since S is an anti-algebra morphism, S^r is also an anti-algebra morphism for all odd r. On the other hand, if r is even, then S^r is an algebra morphism. Then for m an even integer,

$$\begin{aligned} S^m(x) &= S^{m-1}(-g^{n-1}x) \\ &= -S^{m-1}(x)S^{m-1}(g^{n-1}) \\ &= -S^{m-1}(x)(S^{m-1}(g))^{n-1} \\ &= -S^{m-1}(x)(g^{n-1})^{n-1} \\ &= -S^{m-1}(x)g^{n^2-2n+1} \\ &= -S^{m-1}(x)g. \end{aligned}$$

The calculation for m an odd integer is similar. Then we can see recursively that

$$\begin{aligned} S^2(x) &= -S(x)g = -(-g^{n-1}x)g = \lambda x, \\ S^3(x) &= -g^{n-1}S^2(x) = -g^{n-1}(\lambda x) = -\lambda g^{n-1}x \\ S^4(x) &= -S^3(x)g = -(-\lambda g^{n-1}x)g = \lambda^2 x \\ &\vdots \\ S^{2j}(x) &= -S^{2j-1}(x)g = -(-\lambda^{j-1}g^{n-1}x)g = \lambda^j x. \end{aligned}$$

Therefore, the least positive integer m such that $S^m(x) = x$ is $m = 2n$. Of course, this is even, and therefore $S^{2n}(g) = g$ as well. Hence, the order of S is $2n$. ■

2.4 kG, $(kG)^*$, and (co)commutativity

We will look more closely later at Hopf algebras, such as the Taft algebras, that are neither commutative nor cocommutative.

Remark 2.4.1. Notice that the group algebra kG is always cocommutative. Thus, by Proposition 2.2.11 the dual of a group algebra is always commutative. Conversely, it is not difficult to check that kG is a commutative algebra exactly when G is an abelian group. ■

Next, we justify the name "group-like elements" with the following proposition.

Proposition 2.4.2. *Let H be a Hopf algebra. The group-like elements $G(H)$ form a multiplicative group under the multiplication of H.*

Proof. Let M denote the multiplication of H. Certainly the operation is associative since M is associative. Furthermore, $G(H)$ is closed under M, for if $g, h \in G(H)$, then since Δ is an algebra morphism,

$$\begin{aligned}\Delta(gh) &= \Delta(g)\Delta(h)\\ &= (g \otimes g)(h \otimes h)\\ &= (gh) \otimes (gh),\end{aligned}$$

showing $gh \in G(H)$. We already know that for Δ to be an algebra morphism we must have $\Delta(1) = 1 \otimes 1$, so $1 \in G(H)$. Finally, the elements of $G(H)$ are invertible by the antipode:

$$gS(g) = \sum g_1 S(g_2) = \varepsilon(g)1 = 1.$$

Similarly, $S(g)g = 1$. ■

The concept of the group algebra has proved very important in the task of classifying finite dimensional Hopf algebras. In fact, kG is the measure of triviality for Hopf algebras. The following result is a small example.

Proposition 2.4.3. *Let* $\mathrm{ch}(k) \neq 2$ *and* $(H, M, u, \Delta, \varepsilon, S)$ *be a Hopf algebra of dimension two. Then* H *is isomorphic to the group algebra* kC_2*.*

Proof. First, we note that we can write $H = k1 \oplus \mathrm{Ker}(\varepsilon)$. Choose $x \in \mathrm{Ker}(\varepsilon)$ such that $\{1, x\}$ is a basis for H. Then $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ is a basis for $H \otimes H$, so write

$$\Delta(x) = \alpha 1 \otimes 1 + \beta 1 \otimes x + \gamma x \otimes 1 + \delta x \otimes x$$

for some field elements $\alpha, \beta, \gamma, \delta$. Now ε is an algebra morphism, so $\mathrm{Ker}(\varepsilon)$ is a two-sided ideal of the algebra H. Hence $x^2 = ax$ for some $a \in x$. Now, we apply the counit property to x and obtain

$$\begin{aligned} x &= \alpha\varepsilon(1)1 + \beta\varepsilon(1)x + \gamma\varepsilon(x)1 + \delta\varepsilon(x)x \\ &= \alpha 1 + \beta x, \end{aligned}$$

and

$$\begin{aligned} x &= \alpha 1\varepsilon(1) + \beta 1\varepsilon(x) + \gamma x\varepsilon(1) + \delta x\varepsilon(x) \\ &= \alpha 1 + \gamma x. \end{aligned}$$

Therefore, $\alpha = 0$ and $\beta = \gamma = 1$. Now since Δ is an algebra morphism,

$$\begin{aligned} \Delta(x^2) &= \Delta(x)\Delta(x) \\ &= (1 \otimes x + x \otimes 1 + \delta x \otimes x)^2 \\ &= a1 \otimes x + ax \otimes 1 + (2 + 4a\delta + a^2\delta^2)x \otimes x, \end{aligned}$$

where we have used that $x^2 = ax$. On the other hand,

$$\begin{aligned}\Delta(x^2) &= \Delta(ax)\\ &= a\Delta(x)\\ &= a1 \otimes x + ax \otimes 1 + a\delta x \otimes x.\end{aligned}$$

So we obtain the quadratic equation in $a\delta$, $2+4a\delta+a^2\delta^2 = a\delta$, with solutions $a\delta = -1$ and $a\delta = -2$. Now, we apply the antipode property to x, that is, $\sum S(x_1)x_2 = \varepsilon(x)1 = 0$. Recalling that $S(1) = 1$, we have,

$$S(1)x + S(x) + \delta S(x)x = x + S(x)(1 + \delta x) = 0.$$

If we write, $S(x) = c1 + dx$ for some $c, d \in k$, then we have

$$\begin{aligned}0 &= x + (c1 + dx)(1 + \delta x)\\ &= x + c1 + c\delta x + dx + ad\delta x\\ &= c1 + (1 + c\delta + d + da\delta)x.\end{aligned}$$

Immediately, we see that $c = 0$, so we obtain the equation $1 + d + da\delta = 0$. Thus we cannot have $a\delta = -1$, and we must have $a\delta = -2$. Since $\mathrm{ch}(k) \neq 2$, then $a \neq 0$, and $\delta = \frac{-2}{a}$. Finally, we can check that the element $g = 1 + \delta x \in H$ is group-like, since

$$\begin{aligned}\Delta(g) &= \Delta(1 + \delta x)\\ &= \Delta(1) + \delta\Delta(x)\\ &= 1 \otimes 1 + \delta 1 \otimes x + \delta x \otimes 1 + \delta^2 x \otimes x\\ &= 1 \otimes 1 + \delta x \otimes 1 + 1 \otimes \delta x + \delta x \otimes \delta x\\ &= (1 + \delta x) \otimes 1 + (1 + \delta x) \otimes \delta x\\ &= (1 + \delta x) \otimes (1 + \delta x)\\ &= g \otimes g.\end{aligned}$$

Hence, there are two distinct group-like elements, and since group-likes must be linearly independent, a dimension argument yields $H \cong kC_2$. ■

The preceding exercise is a special case of a conjecture made in 1975 by Kaplansky [9] which was proven in 1993 by Zhu [23].

Theorem 2.4.4 (Kaplansky's Conjecture)**.** *Let k be an algebraically closed field of characteristic zero and H a Hopf algebra over k with prime dimension p. Then $H \cong kC_p$.* ■

Remark 2.4.5. Recall that in §2.3 we claimed that Sweedler's 4-dimensional algebra was the smallest non-commutative, non-cocommutative Hopf algebra. The above proposition justifies this claim since if H is a Hopf algebra with dimension two or three, then $H \cong kC_2$ or $H \cong kC_3$, both commutative and cocommutative Hopf algebras. ■

We now give a condition under which a finite dimensional group algebra is self-dual. That is, $kG \cong (kG)^*$. We will need the following lemma:

Lemma 2.4.6. *Let G and H be multiplicative groups. Then $k\{G \times H\} \cong kG \otimes kH$.*

Proof. Define the map $\phi : k\{G \times H\} \to kG \otimes kH$ given by $\phi(g,h) = g \otimes h$. It is easy to check that ϕ is a bijection. We now check that ϕ is a morphism of bialgebras, and therefore a Hopf morphism. Indeed, for $g, g' \in G$ and $h, h' \in H$,

$$\begin{aligned}
\phi((g,h)(g',h')) &= \phi(gg', hh') \\
&= gg' \otimes hh' \\
&= (g \otimes h)(g' \otimes h') \\
&= \phi(g,h)\phi(g',h'),
\end{aligned}$$

showing ϕ is an algebra morphism. Moreover,

$$\begin{aligned}(\phi\otimes\phi)\Delta_{k\{G\times H\}}(g,h) &= (\phi\otimes\phi)((g,h)\otimes(g,h))\\ &= g\otimes h\otimes g\otimes h\\ &= (I\otimes T\otimes I)\circ(\Delta_G\otimes\Delta_H)(g\otimes h)\\ &= \Delta_{kG\otimes kH}(\phi(g,h)),\end{aligned}$$

as desired. ■

Proposition 2.4.7. *Let $n\geq 2$ and k a field containing a primitive n-th root of unity. Let C_n be the cyclic group of order n. Then kC_n is self-dual.*

Proof. Let λ be a primitive n-th root of unity. Denote kC_n by H and write $C_n=\langle c\rangle$. Now, $\{1,c,\ldots,c^{n-1}\}$ forms a basis for H, and let $\{h_1^*,h_c^*,\ldots,h_{c^{n-1}}^*\}$ be the dual basis for H^*. Hence $\dim(H)=\dim(H^*)=n$.

We already know from Proposition 1.5.3 that the group-like elements in H^* are the algebra morphisms. Thus if $f\in G(H^*)$, then $1=f(1)=f(c^n)=(f(c))^n$. Hence, $f(c)=\lambda^i$ for some $0<i<n-1$. On the other hand, if we choose any such i, then there exists a unique algebra morphism, $f_i: H\to k$, given by $f_i(c)=\lambda^i$. In particular, $f_i(c^j)=\lambda^{ij}$, and extended linearly to all of H. Now, the group-like elements, $G(H^*)$ form a multiplicative group by Proposition 2.4.2 and Fact 1.5.2(iii) ensures that they are linearly independent. Thus, after counting dimensions, we see that $H^*=kG(H^*)$.

We have only to show that $G(H^*)\cong C_n$ as a group, that is, $f_i=f_1^i$. The proof is by induction on i. Recall that the multiplication for H^* is given by $m(f\otimes g)(c)=\sum f(c_1)g(c_2)=f(c)g(c)$ since H is a group algebra. Then,

$$f_1^{i+1}(c)=m(f_1^i\otimes f_1)(c)=f_1^i(c)f_1(c)=(\lambda^i)\lambda=\lambda^{i+1}=f_{i+1}(c)$$

Then $G(H^*)\cong C_n$, and therefore, $(kC_n)^*=H^*=kG(H^*)\cong kC_n$. ■

Corollary 2.4.8. *Let G be a finite abelian group. Then kG is self-dual.*

Proof. First we note that $G \cong C_{n_1} \times \cdots \times C_{n_m}$ for some natural numbers $n_1, \ldots, n_m$. We may induct on Lemma 2.4.6 to obtain that $kG \cong kC_{n_1} \otimes \cdots \otimes kC_{n_m}$. Then we may use the isomorphism θ_m defined in Corollary 1.4.3 to get

$$(kG)^* \cong (kC_{n_1} \otimes \cdots \otimes kC_{n_m})^* \cong (kC_{n_1})^* \otimes \cdots \otimes (kC_{n_m})^*.$$

Using the preceding proposition, we obtain

$$(kG)^* \cong (kC_{n_1})^* \otimes \cdots \otimes (kC_{n_m})^* \cong kC_{n_1} \otimes \cdots \otimes kC_{n_m} \cong kG,$$

showing that kG is self-dual. ■

We end this section with an important result in the classification of Hopf algebras. We have already stated that kG is always cocommutative and hence its dual is always commutative. However, the following result from the collective work done by Milnor and Moore [17], Cartier [2], and Kostant [11], gives the converse in the finite dimensional case. That is, it ensures that *all* finite dimensional cocommutative Hopf algebras over an algebraically closed field with characteristic 0 are isomorphic to group algebras.

Theorem 2.4.9 (The Cartier-Kostant-Milnor-Moore Theorem)**.** *A cocommutative Hopf algebra over an algebraically closed field of characteristic zero is a semidirect product of a group algebra and the enveloping algebra of a Lie algebra. In particular, a finite dimensional cocommutative Hopf algebra is a group algebra.* ■

Corollary 2.4.10. *A finite dimensional commutative Hopf algebra over an algebraically closed field of characteristic 0 is the dual of a group algebra.* ■

Corollary 2.4.11. *If H is a finite dimensional commutative, cocommutative Hopf algebra over an algebraically closed field of characteristic 0, then $H \cong (kG)^*$ for an abelian group G.* ■

2.5 Calculation in the dual

The goal of this section is to give some examples of calculations in a dual Hopf algebra, in particular, the dual of a group algebra $(kG)^*$, and the dual of a Taft algebra $H_n(\lambda)^*$. If H is the Hopf algebra in question, we will write, for both H and H^*, the comultiplication as Δ and the counit as ε, since the intention should be clear from the context. In contrast, we will write the antipode of H (resp. H^*) as S (resp. S^* the transpose map of S). We will also need to recall several important results of §1.4, including Fact 1.4.6(i); that is, for any $f \in H^*$ and $a, b \in H$, $f(ab) = \sum f_1(a)f_2(b)$. First we establish a useful lemma.

Lemma 2.5.1. *Let H be a finite dimensional Hopf algebra with basis $\{e_i\}_i$. If the dual basis is written $\{e_i^*\}_i$, then for any $f \in H^*$,*

$$\Delta(f) = \sum_{i,j} f(e_i e_j) e_i^* \otimes e_j^*.$$

Proof. Since $\{e_i^* \otimes e_j^*\}_{i,j}$ is a basis for $H^* \otimes H^*$, we can write

$$\Delta(f) = \sum_{i,j} a_{i,j} e_i^* \otimes e_j^*$$

for some scalars $a_{i,j}$. By Fact 1.4.6 we have

$$f(e_s e_r) = \sum_{i,j} a_{i,j} e_i^*(e_s) e_j^*(e_r) = a_{s,r}.$$

Therefore, $\Delta(f) = \sum_{i,j} f(e_i e_j) e_i^* \otimes e_j^*$, as desired. ■

Proposition 2.5.2 (Calculation of $(kG)^*$)**.** *Let G be a finite group and kG the associated Hopf algebra. Let $\{p_g\}_{g \in G}$ denote the basis for $(kG)^*$ which is dual to the basis $\{g\}_{g \in G}$ for kG. Then the Hopf algebra structure of $(kG)^*$ can be expressed in terms of $\{p_g\}_{g \in G}$ by:*

$$p_g p_h = 0 \textit{ if } g \neq h,\ p_g^2 = p_g,\ \sum_{g \in G} p_g = 1_{(kG)^*}.$$

$$\Delta(p_g) = \sum_{h\in G} p_{gh^{-1}} \otimes p_h = \sum_{h\in G} p_h \otimes p_{h^{-1}g},\ \varepsilon(p_g) = \delta_{1,g}.$$
$$S^*(p_g) = p_{g^{-1}}.$$

Proof. Given the definition for multiplication in the dual of a coalgebra (§1.4), we have for $x \in G$,

$$(p_g * p_h)(x) = \sum p_g(x_1)p_h(x_2) = p_g(x)p_h(x) = \begin{cases} 1 & , \quad g = h = x \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Thus, $p_g p_h = 0$ for $g \neq h$ and $p_g^2 = p_g$. In other words, $\{p_g\}_{g\in G}$ is a collection of orthogonal idempotents. Next, we write $1_{(kG)^*} = \sum_{g\in G} a_g p_g$, for some scalars a_g. We again use the definition for multiplication in the dual and consider, for any $f \in (kG)^*$ and $g \in G$,

$$(1_{(kG)^*} * f)(g) = \sum 1_{(kG)^*}(g_1)f(g_2) = 1_{(kG)^*}(g)f(g) = a_g f(g).$$

Now, if $1_{(kG)^*}$ is to be the identity element of $(kG)^*$ then we must have that $a_g = 1$ for all $g \in G$. Thus, $1_{(kG)^*} = \sum_{g\in G} p_g$.

For the comultiplication, we can begin by using Lemma 2.5.1 to obtain that

$$\Delta(p_g) = \sum_{x,y\in G} p_g(xy)(p_x \otimes p_y).$$

Now, the only nonzero terms above are those for which $xy = g$; that is, $x = gy^{-1}$ or $y = x^{-1}g$, and in this case, $p_g(xy) = 1$. Hence, the comultiplication in $(kG)^*$ is given in terms of the dual basis by

$$\Delta(p_g) = \sum_{x\in G} p_x \otimes p_{x^{-1}g} = \sum_{y\in G} p_{gy^{-1}} \otimes p_y.$$

Alternatively, we could write,

$$\Delta(p_g) = \sum_{xy=g} p_x \otimes p_y.$$

The counit comes directly from the definition given in §1.4, that for any $f \in H^*$, $\varepsilon(f) = f(1)$. Thus $\varepsilon(p_g) = \delta_{1,g}$. The antipode follows from the computation, $S^*(p_g)(h) = p_g(S(h)) = p_g(h^{-1})$ for all $g, h \in G$. ∎

Now we turn our attention to the Taft algebras $H_n(\lambda)$. Recall that these algebras are generated by the pair $\langle g, x\rangle$ with vector space basis given by $\{g^i x^j\}$ for $0 \leq i, j \leq n-1$. We will denote the dual basis in $H_n(\lambda)^*$ by $\{p_{i,j}\}$ where $p_{i,j}(g^k x^\ell) = 1$ if and only if $i = k, j = \ell$, otherwise, $p_{i,j}(g^k x^\ell) = 0$.

Lemma 2.5.3. *For any $f \in H_n(\lambda)^*$,*

$$\Delta(f) = \sum_{i,j,k,\ell=0}^{n-1} \lambda^{jk} f(g^{i+k} x^{j+\ell}) p_{i,j} \otimes p_{k,\ell}$$

Proof. We can immediately write, using Lemma 2.5.1,

$$\Delta(f) = \sum_{i,j,k,\ell=0}^{n-1} f(g^i x^j g^k x^\ell) p_{i,j} \otimes p_{k,\ell}.$$

Using the relations in $H_n(\lambda)$, specifically $xg = \lambda gx$, we have,

$$\begin{aligned} g^i x^j g^k x^\ell &= g^i x^{j-1}(xg) g^{k-1} x^\ell \\ &= g^i x^{j-1}(\lambda gx) g^{k-1} x^\ell \\ &= \lambda g^i x^{j-1} g(xg) g^{k-2} x^\ell \\ &\vdots \\ &= (\lambda^k) g^i x^{j-1} g^k x^{1+\ell} \\ &\vdots \\ &= (\lambda^{jk}) g^{i+k} x^{j+\ell}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta(f) &= \sum_{i,j,k,\ell=0}^{n-1} f(g^i x^j g^k x^\ell) p_{i,j} \otimes p_{k,\ell} \\ &= \sum_{i,j,k,\ell=0}^{n-1} f(\lambda^{jk} g^{i+k} x^{j+\ell}) p_{i,j} \otimes p_{k,\ell} \\ &= \sum_{i,j,k,\ell=0}^{n-1} \lambda^{jk} f(g^{i+k} x^{j+\ell}) p_{i,j} \otimes p_{k,\ell}, \end{aligned}$$

as desired. ∎

Theorem 2.5.4. *The Sweedler algebra $H_2(-1)$ is self-dual.*

Proof. It is possible to establish this result by showing that the Sweedler algebra is the unique non-commutative, non-cocommutative Hopf algebra of dimension four and hence must be self-dual, but here we calculate the dual explicitly for emphasis. We will take as a basis that which is dual to $\{1, g, x, gx\}$, the standard basis in $H_2(-1)$. We will denote this basis by $\{p_1, p_g, p_x, p_{gx}\}$. Notice that as a special case of the Taft algebra ($n = 2$), this set corresponds to the notation above for the dual basis when written as $\{p_{0,0}, p_{1,0}, p_{0,1}, p_{1,1}\}$, respectively. We begin by using Lemma 2.5.3 to calculate the comultiplication for each of these elements.

Comultiplication of p_1: By 2.5.3, we can write

$$\Delta(p_1) = \sum_{i,j,k,\ell=0}^{1} (-1)^{jk} p_1(g^{i+k}x^{j+\ell})p_{i,j} \otimes p_{k,\ell}.$$

Using the lemmas to aid calculation, one can check that

$$\Delta(p_1) = p_1 \otimes p_1 + p_g \otimes p_g. \tag{2.5}$$

Comultiplication of p_g: Similarly, some computation gives

$$\Delta(p_g) = p_1 \otimes p_g + p_g \otimes p_1 \tag{2.6}$$

Comultiplication of p_x: Using Lemma 2.5.3, the only terms which are nonzero have $i + k = 0 \mod 2$ and $j + \ell = 1$. Notice we require the sum $j + \ell$ to be *exactly* one since it represents an exponent on the element $x \in H_2(-1)$. If this exceeds one, then the resulting product $g^{i+k}x^{j+\ell} = 0$ for all i, k. This results in the comultiplication,

$$\Delta(p_x) = p_1 \otimes p_x + p_g \otimes p_{gx} + p_x \otimes p_1 - p_{gx} \otimes p_g. \tag{2.7}$$

Comultiplication of p_{gx}: Similarly, we have that

$$\Delta(p_{gx}) = p_1 \otimes p_{gx} + p_g \otimes p_x - p_x \otimes p_g + p_{gx} \otimes p_1. \tag{2.8}$$

Using the definition of the counit in a dual Hopf algebra, it is easy to see that $\varepsilon(p_1) = 1$, while $\varepsilon(p_g) = \varepsilon(p_x) = \varepsilon(p_{gx}) = 0$.

Next, we seek a multiplication table in $H_2(-1)^*$ in terms of $\{p_1, p_g, p_x, p_{gx}\}$. We use the fact that for any $\alpha, \beta \in H^*$ and $h \in H$, the relation $(\alpha * \beta)(h) = \sum \alpha(h_1)\beta(h_2)$ always holds. We know that in $H_2(-1)^*$, we can write any such product as

$$\alpha\beta = c_1^{(\alpha\beta)} p_1 + c_g^{(\alpha\beta)} p_g + c_x^{(\alpha\beta)} p_x + c_{gx}^{(\alpha\beta)} p_{gx},$$

for some scalars $c_1^{(\alpha\beta)}, c_g^{(\alpha\beta)}, c_x^{(\alpha\beta)}, c_{gx}^{(\alpha\beta)}$. Consider the following example calculation:

$$(p_g p_x)(x) = p_g(g)p_x(x) + p_g(x)p_x(1) = 1, \quad \Rightarrow c_x^{(p_g p_x)} = 1.$$

Similar computations yield the following multiplication table, with products written as “element from left column” times “element from top row”:

$*$	p_1	p_g	p_x	p_{gx}
p_1	p_1			p_{gx}
p_g		p_g	p_x	
p_x	p_x			
p_{gx}		p_{gx}		

where blank cells are understood to be zero. We are now in a position to see that $H_2(-1)^* \cong H_2(-1)$. Indeed, if we set

$$1^* = p_1 + p_g,$$

$$g^* = p_1 - p_g,$$

$$x^* = p_x - p_{gx},$$

we can check that these elements mimic the roles of $1, g, x \in H_2(-1)$, respectively. Using the multiplication table above, we can immediately see that g^*x^*, the element that should mimic $gx \in H_2(-1)$, can be written as $g^*x^* = -(p_x + p_{gx})$. It is easy to check that the set $\{1^*, g^*, x^*, g^*x^*\}$ is a basis for $H_2(-1)^*$. Next, we verify each of the algebraic relations. Let $f \in H_2(-1)^*$, then for scalars $\alpha, \beta, \gamma, \delta$, let $h =$

$\alpha 1 + \beta g + \gamma x + \delta gx$. Consider

$$
\begin{aligned}
(1^* * f)(h) &= \sum 1^*(h_1) f(h_2) \\
&= 1^*(1) f(\alpha 1) + 1^*(g) f(\beta g) + 1^*(g) f(\gamma x) \\
&\quad + 1^*(x) f(\gamma 1) + 1^*(1) f(\delta gx) + 1^*(gx) f(\delta g) \\
&= f(\alpha 1) + f(\beta g) + f(\gamma x) + 0 + f(\delta gx) + 0 \\
&= f(\alpha 1 + \beta g + \gamma x + \delta gx) \\
&= f(h).
\end{aligned}
$$

Similarly, $(f * 1^*)(h) = f(h)$. Hence, 1^* is the identity of $H_2(-1)^*$. Now, using the multiplication table, consider

$$
\begin{aligned}
&(g^*)^2 = (p_1 - p_g)^2 = p_1^2 - p_1 p_g - p_g p_1 + p_g^2 = p_1 + p_g = 1^*, \text{ and} \\
&(x^*)^2 = (p_x - p_{gx})^2 = p_x^2 - p_x p_{gx} - p_{gx} p_x + p_{gx}^2 = 0.
\end{aligned}
$$

So g^* squares to the identity and x^* squares to 0, as desired. Moreover,

$$
\begin{aligned}
x^* g^* &= (p_x - p_{gx})(p_1 - p_g) \\
&= p_x p_1 - p_x p_g - p_{gx} p_1 + p_{gx} p_g \\
&= p_x + 0 + 0 + p_{gx} \\
&= -g^* x^*.
\end{aligned}
$$

So $H_2(-1)^* \cong H_2(-1)$ as an algebra. To check the coalgebra structure, we use equations (2.5) and (2.6) to see

$$
\begin{aligned}
\Delta(g^*) &= \Delta(p_1) - \Delta(p_g) \\
&= p_1 \otimes p_1 + p_g \otimes p_g - (p_1 \otimes p_g + p_g \otimes p_1) \\
&= p_1 \otimes (p_1 - p_g) - p_g \otimes (p_1 - p_g) \\
&= (p_1 - p_g) \otimes (p_1 - p_g) \\
&= g^* \otimes g^*,
\end{aligned}
$$

and of course, $\varepsilon(g^*) = g^*(1) = 1$. From equations (2.7) and (2.8) we observe that

$$\begin{aligned}
\Delta(x^*) &= \Delta(p_x) - \Delta(p_{gx}) \\
&= p_1 \otimes p_x + p_g \otimes p_{gx} + p_x \otimes p_1 - p_{gx} \otimes p_g \\
&\quad - (p_1 \otimes p_{gx} + p_g \otimes p_x - p_x \otimes p_g + p_{gx} \otimes p_1) \\
&= p_1 \otimes (p_x - p_{gx}) - p_g \otimes (p_x - p_{gx}) \\
&\quad + p_x \otimes (p_1 + p_g) - p_{gx} \otimes (p_1 + p_g) \\
&= (p_1 - p_g) \otimes (p_x - p_{gx}) + (p_x - p_{gx}) \otimes (p_1 + p_g) \\
&= g^* \otimes x^* + x^* \otimes 1^*,
\end{aligned}$$

and furthermore, $\varepsilon(x^*) = x^*(1) = 0$, as desired. Finally, we check the antipodes. Again, consider the general element $h = \alpha 1 + \beta g + \gamma x + \delta gx$. Then

$$S^*(g^*)(h) = g^*(S(h)) = g^*(\alpha 1 + \beta g - \gamma gx + \delta x) = \alpha - \beta.$$

Hence $S^*(g^*) = p_1 - p_g = g^*$. For x^*, we have that

$$S^*(x^*)(h) = x^*(S(h)) = x^*(\alpha 1 + \beta g - \gamma gx + \delta x) = \gamma + \delta.$$

Thus, $S^*(x^*) = p_x + p_{gx} = -g^*x^*$, and the verification is complete. ■

What, if any, of the computations above can be generalized to the duals of the larger Taft algebras, $H_n(\lambda)^*$ $(n > 2)$? As before, we will attempt to compute elements $g^*, x^* \in H_n(\lambda)^*$ which correspond exactly to the generators $\langle g, x\rangle$ of $H_n(\lambda)$, giving rise to the same Hopf algebra structure. Recall that we will write the basis dual to $\{g^i x^j\}$, $0 \leq i, j \leq n-1$, as $\{p_{i,j}\}$. We have already calculated a formula for the comultiplication in terms of this basis in Lemma 2.5.3. We begin much as we did for the Sweedler algebra, and explicitly write the comultiplication for each $p_{i,j}$ with $j = 0$. That is, the comultiplication for dual basis elements corresponding to the group-like elements $\{1, g, g^2, \ldots, g^{n-1}\}$ in $H_n(\lambda)$.

The formula in Lemma 2.5.3 now says that the nonzero terms in $\Delta(p_{0,0})$ are those in which $i+k=0 \mod n$, and $j+\ell=0$. There are n possible cases, one for each of the values $0 \leq i \leq n-1$, and hence we have

$$\Delta(p_{0,0}) = p_{0,0} \otimes p_{0,0} + p_{1,0} \otimes p_{n-1,0} + p_{2,0} \otimes p_{n-2,0} + \cdots + p_{n-1,0} \otimes p_{1,0}. \tag{2.9}$$

In general, to determine $\Delta(p_{r,0})$ $(0 \leq r \leq n-1)$, the process is the same, and we have that

$$\Delta(p_{r,0}) = \sum p_{s,0} \otimes p_{t,0}, \tag{2.10}$$

where the sum is taken over all pairs (s,t) such that $0 \leq s,t \leq n-1$ and $s+t=r \mod n$. Just as in the Sweedler algebra, each coefficient is known to be 1 since $j=0 \Rightarrow \lambda^{jk}=1$. In terms of multiplication, we can check, using similar methods to those used for the Sweedler algebra, that the following relations hold for the elements $\{p_{i,0}\}$:

$$p_{i,0} * p_{k,0} = \delta_{i,k}, \quad \sum_{i=0}^{n-1} p_{i,0} = 1_{H_n(\lambda)^*}. \tag{2.11}$$

We are now in a position to prove an interesting proposition regarding the group-like elements in $H_n(\lambda)^*$. This is a first step toward comparing $H_n(\lambda)$ and $H_n(\lambda)^*$, implying that, at least their group-like structures are isomorphic.

Proposition 2.5.5. *$G(H_n(\lambda)^*) \cong C_n$.*

Proof. Recall that if $f \in G(H_n(\lambda)^*)$ then f is an algebra morphism. Therefore, writing $f = \sum_{i,j} a_{i,j} p_{i,j}$, for some scalars $a_{i,j}$, we can check that $1 = f(1) = a_{0,0}$, and moreover $1 = f(g^n) = f(g)^n = (a_{1,0})^n$. Hence, $a_{1,0} = \xi$, some n-th root of unity. In addition, we have the relations:

$$\begin{aligned} a_{2,0} &= f(g^2) = f(g)^2 = (a_{1,0})^2, \\ a_{3,0} &= f(g^3) = f(g)^3 = (a_{1,0})^3, \\ &\vdots \\ a_{n-1,0} &= f(g^{n-1}) = f(g)^{n-1} = (a_{1,0})^{n-1}. \end{aligned}$$

We also have $0 = f(0) = f(x^n) = f(x)^n = (a_{0,1})^n$, so $a_{0,1} = 0$, and the relations

$$\begin{aligned} a_{0,2} = f(x^2) &= f(x)^2 = (a_{0,1})^2 = 0, \\ a_{0,3} = f(x^3) &= f(x)^3 = (a_{0,1})^3 = 0, \\ &\vdots \\ a_{0,n-1} = f(x^{n-1}) &= f(x)^{n-1} = (a_{0,1})^{n-1} = 0. \end{aligned}$$

Hence for all $j \neq 0$, $a_{i,j} = f(g^i x^j) = f(g^i)f(x^j) = a_{i,0}a_{0,j} = 0$. Therefore,

$$f = p_{0,0} + \xi p_{1,0} + \xi^2 p_{2,0} + \cdots + \xi^{n-1} p_{n-1,0}.$$

Now λ is a primitive n-th root of unity, so the element $g^* = p_{0,0} + \lambda p_{1,0} + \lambda^2 p_{2,0} + \cdots + \lambda^{n-1} p_{n-1,0}$ can generate all such $f \in G(H_n(\lambda)^*)$. Indeed, since λ is primitive, there must exist $m < n$ such that $\lambda^m = \xi$. Using the multiplicative relations in (2.11), we observe that

$$\begin{aligned} (g^*)^m &= (p_{0,0} + \lambda p_{1,0} + \lambda^2 p_{2,0} + \cdots + \lambda^{n-1} p_{n-1,0})^m \\ &= p_{0,0} + (\lambda^m) p_{1,0} + (\lambda^m)^2 p_{2,0} + \cdots + (\lambda^m)^{n-1} p_{n-1,0} \\ &= p_{0,0} + \xi p_{1,0} + \xi^2 p_{2,0} + \cdots + \xi^{n-1} p_{n-1,0} \\ &= f. \end{aligned}$$

Again we use (2.11):

$$\begin{aligned} (g^*)^n &= (p_{0,0} + \lambda p_{1,0} + \lambda^2 p_{2,0} + \cdots + \lambda^{n-1} p_{n-1,0})^n \\ &= p_{0,0} + (\lambda^n) p_{1,0} + (\lambda^n)^2 p_{2,0} + \cdots + (\lambda^n)^{n-1} p_{n-1,0} \\ &= p_{0,0} + p_{1,0} + p_{2,0} + \cdots + p_{n-1,0} \\ &= 1_{H_n(\lambda)^*}. \end{aligned}$$

Therefore, $G(H_n(\lambda)^*) = \langle g^* \rangle \cong C_n$. ■

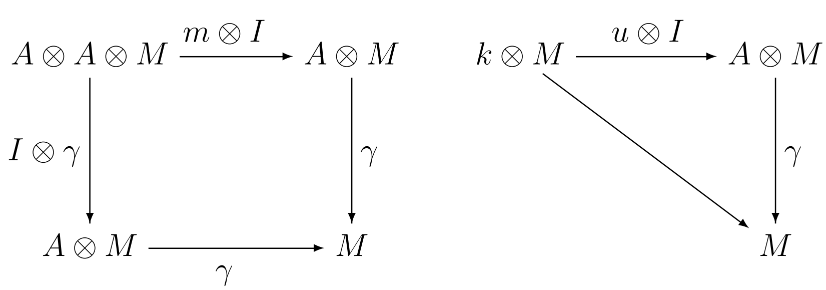
Chapter 3: Hopf algebra actions on rings

In practice, we have demonstrated in §2.4 that when working over an algebraically closed field of characteristic zero, to study finite dimensional cocommutative (resp. commutative) Hopf algebras is to study group algebras kG (resp. duals of group algebras $(kG)^*$). As a consequence, we will see that to generate the most general actions on algebras, we will have to study Hopf algebras that are both non-commutative and non-cocommutative. In fact, we will refer to any finite dimensional Hopf algebra which is either commutative or cocommutative as a **trivial** Hopf algebra. We begin with some definitions.

3.1 Definitions and Examples

Throughout this chapter, H is a Hopf algebra over the field k with comultiplication Δ, counit ε, and antipode S. A is a k-algebra. First we recall the definition of a module.

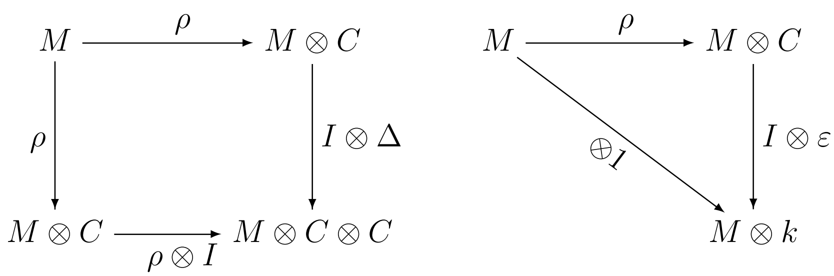
Definition 3.1.1. *Let (A, m, u) be a k-algebra. Then a* **left A-module** *is a k-space M with a k-linear map $\gamma : A \otimes M \to M$ such that the following diagrams commute:*



where I denotes the identity map and the unadorned diagonal arrow indicates scalar multiplication.

Just as algebras and coalgebras are dual constructions, there is an analogous dualized structure for modules.

Definition 3.1.2. *Let* (C, Δ, ε) *be a* k*-coalgebra. Then a* **right** C**-comodule** *is a* k*-space* M *with a* k*-linear map* $\rho : M \to M \otimes C$ *such that the following diagrams commute:*



where I *denotes the identity map.*

The maps in a comodule are defined in the obvious way:

Definition 3.1.3. *Let* (M, ρ_M) *and* (N, ρ_N) *be right* C*-comodules and* $f : M \to N$*.* f *is called a* **comodule map** *if* $\rho_N \circ f = (f \otimes I) \circ \rho_M$*.*

Now we can define a Hopf algebra action.

Definition 3.1.4. *A* k*-algebra* A *is a* **left** H**-module algebra** *if* A *is a left* H*-module and the following hold:*

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \tag{3.1}$$

$$h \cdot 1_A = \varepsilon(h) 1_A \tag{3.2}$$

for all $a, b \in A$ *and* $h \in H$*, where we have denoted the action of the element* $h \in H$ *on the element* $a \in A$ *by* $h \cdot a$*. In this case, we say that* H *acts on* A*.*

Right H-module algebras can be defined analogously, but we will primarily be concerned with left actions, so henceforth, the phrase "H acts on A" should be interpreted as the left action as given above, unless explicitly stated otherwise.

Definition 3.1.5. *An algebra (A, M, u) is called a* **right H-comodule algebra** *if*

(i) A is a right H-comodule

(ii) M and u are right H-comodule maps.

Proposition 3.1.6. *Let H be a finite-dimensional Hopf algebra. Then A is a left H-module algebra if and only if A is a right H^*-comodule algebra.*

Proof. See Montgomery [18]. ■

Armed with a definition for Hopf actions, we turn our attention to the major goal of this thesis. We will demonstrate that Hopf algebras are natural candidates to replace finite groups as *actors* on rings. Moreover, we wish to ask what remains invariant under such an action and will compute several specific examples. Before this however, we will see that we need an updated idea of invariance.

Traditionally, if an element a is invariant under an action, we require that $h \cdot a = a$ for all $h \in H$. However, in the case of Hopf algebras, problems arise. For a simple example, suppose $a, b \in A$ both have the property that $h \cdot a = a$ and $h \cdot b = b$ for all $h \in H$. Further, suppose that H contains, for instance, a primitive element x. That is, $\Delta(x) = 1 \otimes x + x \otimes 1$. Then consider,

$$x \cdot (ab) = (1 \cdot a)(x \cdot b) + (x \cdot a)(1 \cdot b) = (a)(b) + (a)(b) = 2ab.$$

Although a and b are both "fixed" under the action of H, the same is not necessarily true of their product. Hence, the set $\{a \in A \,|\, h \cdot a = a \ \forall h \in H\}$ may not be a subalgebra of A. We wish to preserve this property, after all it is true for group actions on rings, which motivates the following alternative definition of the invariant algebra.

Definition 3.1.7. *Let A be an H-module algebra. Then the* **algebra of invariants** *under the action of H is the set*

$$A^H = \{a \in A \mid h \cdot a = \varepsilon(h)a, \ \forall h \in H\}.$$

We may alternatively refer to this set as the **fixed algebra**.

Notice that this definition is consistent with the traditional one whenever $H = kG$ since $\varepsilon(g) = 1$ for all $g \in G$. Moreover, this will solve the dilemma in the example above since $x \in P(H)$ guarantees that $\varepsilon(x) = 0$. Thus, this definition has the surprising consequence that a primitive element (or any other element in $\operatorname{Ker}\varepsilon$) has the property that it "fixes" only those elements which it sends to zero. Though this may at first seem bizarre, Definition 3.1.7 is the correct generalization, since it indeed produces a subalgebra.

Proposition 3.1.8. *Let A be an H-module algebra. Then A^H is a subalgebra of A.*

Proof. The subspace condition is easily checked. The property 3.2 insures that $1_A \in A^H$. Certainly the multiplication is associative. We must show that the multiplication is closed. Take any $a, b \in A^H$. Then

$$\begin{aligned} h \cdot (ab) &= \sum (h_1 \cdot a)(h_2 \cdot b) \\ &= \sum (\varepsilon(h_1)a)(\varepsilon(h_2)b) \\ &= \sum \varepsilon(h_1)\varepsilon(h_2)(ab) \\ &= \varepsilon\left(\sum \varepsilon(h_1)h_2\right)(ab), \quad \text{(linearity of } \varepsilon) \\ &= \varepsilon(h)(ab), \quad \text{(counit property)} \end{aligned}$$

showing that $ab \in A^H$. ■

Sometimes we will want to talk about the elements of A invariant under only a single element $h \in H$. We will denote this set by $A^h = \{a \in A \,|\, h \cdot a = \varepsilon(h)a\}$. By the same process as above, one can check that A^h is a subalgebra of A. Now we consider actions of our prototypical examples, kG and $(kG)^*$.

Example 3.1.9. Let $H = kG$ and suppose that H acts on A. Since $\Delta(g) = g \otimes g$, $g \cdot (ab) = (g \cdot a)(g \cdot b)$ for all $a, b \in A$, $g \in G$. Moreover, if $a \in A^H$, then $g \cdot a = \varepsilon(g)a = a$ for all $g \in H$. Thus the action of a group algebra mimics the action of groups acting as automorphisms. More generally, we can say that for any H-module algebra A, the set of group-like elements $G(H)$ acts like a group of automorphisms on A. ■

Example 3.1.10. Let G be a finite group and A an algebra graded by G. Then, write $A = \bigoplus_{g \in G} A_g$, such that $A_g A_h \subseteq A_{gh}$, $\forall g, h \in G$ and in particular $1_A \in A_{1_G}$. Now, any $a \in A$ can be written as $a = \sum_{g \in G} a_g$, where we let a_g denote the g-component of a. Let $H = (kG)^*$ and denote the dual basis by $\{p_g\}_{g \in G}$. It is not difficult to check that the action $p_g \cdot a = a_g$ for all $g \in G$ makes A an H-module. Using the grading and the comultiplication in $(kG)^*$ calculated in Proposition 2.5.2, we see that for any $g \in G$,

$$p_g \cdot (ab) = \sum_{xy=g} p_x(a) p_y(b) = \sum_{xy=g} (a_x)(b_y).$$

On the other hand,

$$(ab)_g = \left(\left(\sum_{x \in G} a_x \right) \left(\sum_{y \in G} b_y \right) \right)_g = \sum_{xy=g} (a_x)(b_y),$$

which establishes property (3.1) in Definition 3.1.4. Furthermore, for any grading of A, we must have that $(1_A)_g = \delta_{1,g}$, satisfying property (3.2).

Conversely, suppose that A is a left $(kG)^*$-module algebra. Alternatively, this means that A is a right (kG)-comodule algebra with some structure map ρ. We will begin by writing $\rho(a) = \sum_g a_g \otimes g$ and attempt to calculate the "coefficients" a_g. Using the fact that A is a comodule we have that $(\rho \otimes I) \circ \rho = (I \otimes \Delta) \circ \rho$ and thus we have $(a_g)_h = \delta_{g,h} a_g$. Therefore, $\rho(a_g) = a_g \otimes g$. Then for each g, define the subspace $A_g = \{a_g \mid a \in A\}$ and by our computation we have that $A_g \cap A_h = 0$ for $g \neq h$. Moreover, the comodule structure requires that for all $a \in A$ we have $a \otimes 1 = (I \otimes \varepsilon)\rho(a)$. Thus, $\sum_g a_g = a$ for all $a \in A$. Therefore we may write $A = \bigoplus_{g \in G} A_g$ so that A is a G-graded vector space. Furthermore, this decomposition provides an algebra grading. Indeed, let $a_g \in A_g$ and $b_h \in A_h$ and recall that the multiplication of A must be a comodule map to see that $\rho(a_g b_h) = \rho(a_g)\rho(b_h) = a_g b_h \otimes gh$. That is, $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. Similarly, since u is a comodule map, $1_A = u(1) \in A_{1_G}$. ∎

Remark 3.1.11. The importance of the two previous examples cannot be overstated. Recall that over algebraically closed fields of characteristic 0, if H is cocommutative, $H \cong kG$. On the other hand, when H is commutative, $H \cong (kG)^*$. Hence, when working over algebraically closed fields of characteristic 0, to study actions of cocommutative Hopf algebras is to study group actions. Similarly, to study actions of commutative Hopf algebras is to study G-graded algebras for finite groups G. ∎

In the sections that follow, we will introduce some specific examples of actions of Hopf algebras which are both non-commutative and non-cocommutative on various rings. In terms of the algebras being acted upon, we will be most interested in two specific examples: polynomial rings $k[x_1, \ldots, x_n]$ and the so-called **quantum plane**, $k_q[x, y] = k\langle x, y \mid yx - qxy\rangle$. In particular, both of these algebras share the property that they are graded by degree. That is, they can be written as $A = \bigoplus_n A_n$ where each A_n is the vector space of all monomials of degree n and $A_i A_j \subset A_{i+j}$. Additionally, all of the actions we consider will **preserve degree**. That is, if $h \in H$ and $a_i \in A_i$, then $h \cdot a_i \in A_i$. The following propositions deal with matrix representations for automorphisms on these algebras, and will prove useful as we progress.

Proposition 3.1.12. *Let $A = k[x_1, \ldots, x_n]$, a polynomial ring and suppose H acts on A. Conjugation in the representation of H induces an automorphism of A.*

Proof. The result follows since conjugation is accomplished by multiplication of an invertible matrix. Any such matrix represents an automorphism of A. ■

Proposition 3.1.13. *Let $q = -1$ and $A = k_q[x, y]$. Suppose σ acts as a degree preserving automorphism of A. Then the 2×2 matrix representing σ as a transformation on A_1 must be either a diagonal or skew diagonal matrix.*

Proof. First write

$$\sigma \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Now σ is an automorphism, so it must be invertible, thus $ad \neq bc$. Moreover

$$\sigma \cdot xy = (\sigma \cdot x)(\sigma \cdot y) = (ax + cy)(bx + dy) = abx^2 + (ad - bc)xy + cdy^2,$$

and

$$\sigma \cdot yx = (\sigma \cdot y)(\sigma \cdot x) = (bx + dy)(ax + cy) = abx^2 + (bc - ad)xy + cdy^2.$$

From the relation $\sigma(yx) = -\sigma(xy)$, we obtain the equations $ab = 0$ and $cd = 0$. Moreover, from $ad \neq bc$, we note that we cannot have both $a = c = 0$. Similarly, we cannot have both $b = d = 0$. It is not difficult to check that if $b = 0$ we obtain a diagonal representation and when $b \neq 0$ we obtain a skew diagonal representation. ■

Proposition 3.1.14. *Let $A = k_q[x, y]$ and $q^2 \neq 1$. If σ is a degree preserving automorphism of A, then σ must be represented in 2×2 matrices by a diagonal matrix.*

Proof. As before, begin by letting

$$\sigma \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

such that the above matrix has full rank, that is $ad \neq bc$. Consider,

$$\sigma \cdot xy = (\sigma \cdot x)(\sigma \cdot y) = (ax + cy)(bx + dy) = abx^2 + (ad + qbc)xy + cdy^2,$$

and

$$\sigma \cdot yx = (\sigma \cdot y)(\sigma \cdot x) = (bx + dy)(ax + cy) = abx^2 + (bc + qad)xy + cdy^2.$$

We use the relation $\sigma \cdot yx = q(\sigma \cdot xy)$ and equate coefficients to obtain the following system of equations:

$$0 = ab(1 - q)$$
$$0 = cd(1 - q)$$
$$0 = bc(1 - q^2).$$

Some computation gives that the only solution is $b = c = 0$, implying σ is represented by a diagonal matrix. ■

Lemma 3.1.15. *Any matrix over $\mathbb{C}$ with finite order is diagonalizable.*

Proof. Let M be such a matrix and suppose that $M^n = I$. Then the minimal polynomial for M divides $z^n - 1$, a polynomial with distinct roots over $\mathbb{C}$. ■

We conclude this section with another important idea that will be utilized throughout.

Definition 3.1.16. *Let A be an H-module algebra and $I \subseteq H$ a Hopf ideal. The action of H on A is said to be* **factorable through** *I if it induces an action of H/I via $\overline{h} \cdot a = h \cdot a$ such that A is an H/I-module algebra.*

Proposition 3.1.17. *Let A be an H-module algebra and $I \subseteq H$ a Hopf ideal. Then the action of H on A is factorable through I if and only if for every $a \in A$, $h \in I$, $h \cdot a = 0$.*

Proof. Suppose the action is factorable through I. Then if $h \in I$, $0 = \overline{h} \in H/I$. The module condition implies that for any $a \in A$, $0 = 0 \cdot a = \overline{h} \cdot a = h \cdot a$. Conversely, suppose that $h \cdot a = 0$ for all $h \in I$ and $a \in A$. We wish to show that the action

$\overline{h} \cdot a = h \cdot a$ endows A with the structure of an H/I-module algebra. The module condition is not difficult to check and when $h \notin I$, the other conditions are easily established as well. We must verify that equations (3.1) and (3.2) hold in the case that $h \in I$. As above, since $h \in I$, $\overline{h} = 0$ so that $\overline{h} \cdot a = 0$ for all $a \in A$. For any $a, b \in A$ we must have,

$$\sum (\overline{h_1} \cdot a)(\overline{h_2} \cdot b) = \sum (h_1 \cdot a)(h_2 \cdot b) = 0,$$

by definition of the comultiplication in H/I and the fact that either $h_1 \in I$ or $h_2 \in I$ for each term in the sum above. (recall $\Delta(h) \in I \otimes H + H \otimes I$ for a Hopf ideal). This is consistent with $0 = \overline{h} \cdot ab$ and thus establishes equation (3.1). Moreover,

$$\overline{\varepsilon}(\overline{h})1 = \varepsilon(h)1 = 0,$$

again by the definition of $\overline{\varepsilon}$ and the requirement that $\varepsilon(I) = 0$ for a Hopf ideal. This coincides with the condition $\overline{h} \cdot 1 = 0$ and therefore equation (3.2) is established. ■

Remark 3.1.18. Suppose that an element $h \in H$ generates a Hopf ideal. Now imagine that we find a representation $\rho : H \to M(k)$ into an appropriate matrix ring so that ρ gives rise to an action on the degree graded algebra $A = \bigoplus_n A_n$ and furthermore $\rho(h)$ acts as the zero map on all degrees. That is, $h \cdot A_n = 0$ for all n. The preceding proposition says that the action factors through $I = \langle h \rangle$ and we may as well regard the action as simply that of H/I. As we progress, we will want to consider actions of Hopf algebras with a specified dimension. Hence, we will not consider representations giving rise to actions that factor since H/I has dimension strictly less than that of H. ■

In Chapter 4 we will prove that there are no actions by nontrivial semisimple Hopf algebras with dimension less than or equal to 15 on polynomials. However, if we relax the condition that the Hopf algebra be semisimple we can construct an example of a polynomial H-module algebra. The rest of this chapter is dedicated to this example.

3.2 Actions of the Taft algebras

Motivated by the comments of Remark 3.1.11 we turn our attention to the actions of nontrivial Hopf algebras, that is, those that are both non-commutative and non-cocommutative. The Taft algebras $H_n(\lambda)$ provided our first examples of such Hopf algebras and we will exhaust the study of their actions on polynomial rings here. Notice that $H_n(\lambda)$ is not semisimple since the ideal generated by x is nilpotent. From this point forward, we will assume that our field k is algebraically closed of characteristic zero. In fact, unless explicitly stated otherwise, we assume $k = \mathbb{C}$.

3.2.1 Actions on $k[u, v]$

We begin with Hopf algebra actions for $H = H_n(\lambda)$ on polynomials in two variables; that is, $A = k[u, v]$. We seek a representation of $H_n(\lambda)$ as an algebra in $M_2(\mathbb{C})$. Recall from Example 2.3.5 that we must satisfy the following relations:

$$g^n = 1, \quad x^n = 0, \quad xg = \lambda gx, \tag{3.3}$$

where λ is a primitive n-th root of 1. We begin with a lemma.

Lemma 3.2.1. *$\langle x \rangle$ is a Hopf ideal.*

Proof. $\langle x \rangle$ is a coideal since $\varepsilon(x) = 0$ and $\Delta(x) = g \otimes x + x \otimes 1 \in H_n(\lambda) \otimes \langle x \rangle + \langle x \rangle \otimes H_n(\lambda)$. Moreover, $S(\langle x \rangle) \subseteq \langle x \rangle$ since $S(x) = -g^{n-1}x \in \langle x \rangle$. ∎

It is not difficult to show that if x acts as the zero map on degree one monomials, i.e. it is represented by the zero matrix, then by induction it acts as the zero map on all of A. This will cause the action of H to factor, and by the rationale of Remark 3.1.18 we will not consider such representations. Recall that since A is a polynomial algebra, by Proposition 3.1.12, we may make a change of variables in A so that x is in its Jordan form. Now x is nilpotent, but is not the zero matrix, so it must have Jordan form

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, writing $g \mapsto (g_{ij})$ in this basis and using the relations in (3.3), we obtain the following equations:

$$g_{22} = \lambda g_{11}$$
$$g_{21} = 0.$$

Thus, we can write

$$g \mapsto \begin{pmatrix} \alpha & \beta \\ 0 & \lambda\alpha \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{C}$. We must now ensure that $H_n(\lambda)$ can act as a Hopf algebra on $A = k[u, v]$. It is sufficient to check the generators for both $H_n(\lambda)$ and A. That is, it is sufficient to guarantee that $x \cdot (uv) = x \cdot (vu)$ and $g \cdot (uv) = g \cdot (vu)$. The matrix for g is invertible provided $\alpha \neq 0$ so that g, a group-like element, is in fact an automorphism. From this, it is clear the relation for g holds. Consider the relation for x. Given the coalgebra structure in Example 2.3.5, we require that

$$x \cdot (uv) = (g \cdot u)(x \cdot v) + (x \cdot u)v = \alpha u^2,$$

and

$$x \cdot (vu) = (g \cdot v)(x \cdot u) + (x \cdot v)u = u^2$$

be equal, from which we obtain that $\alpha = 1$. Thus the following representation for $H_n(\lambda)$ gives a Hopf algebra action on $A = k[u, v]$:

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 1 & \beta \\ 0 & \lambda \end{pmatrix},$$

for some $\beta \in \mathbb{C}$. Suppose that $\beta \neq 0$. The eigenvalues of the matrix representation for g are 1 and λ, with corresponding eigenvectors $\langle 1, 0 \rangle$ and $\langle \frac{\beta}{\lambda - 1}, \ 1 \rangle$, respectively. Hence, we can make another change of variables and write

$$A = k[u, v] \cong k[u, \ \frac{\beta}{\lambda - 1} u + v],$$

in the new eigenbasis. Of course, with these generators, g is represented by a diagonal matrix,

$$\begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}.$$

Moreover, the representation for x in this basis is the same,

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

since $x \cdot u = 0$ and $x \cdot ((\beta/(\lambda - 1))u + v) = u$. Therefore, we need only to consider the case when $\beta = 0$.

We now calculate the fixed algebra. If we denote the set of elements fixed by g as A^g, then it is clear that $A^{H_n(\lambda)} \subseteq A^g$. We see immediately that $u \in A^g$ since $g \cdot u = u = \varepsilon(g)u$. Moreover, we notice that for $i \in \mathbb{N}$ that $g \cdot (v^i) = g \cdot (v^{i-1})(g \cdot v)$ and by induction on i, we obtain that $g \cdot (v^i) = (g \cdot v)^i$. Therefore, $g \cdot (v^i) = (\lambda v)^i$, so that if $i = n$, $g \cdot (v^n) = \varepsilon(g)v^n$. Hence, $A^g = k[u, v^n]$.

Now, we need to find all elements of $k[u, v^n]$ which are also fixed by x, that is, such that $x \cdot a = \varepsilon(x)a = 0$. It is immediately clear that $x \cdot u = 0$, but again we must consider powers of v. Note that $x \cdot (v^2) = (g \cdot v)(x \cdot v) + (x \cdot v)v = \lambda uv + uv = (\lambda + 1)uv$. By induction on ℓ, we can show:

Lemma 3.2.2. *The following relation holds for all $\ell \in \mathbb{N}$.*

$$x \cdot (v^\ell) = (\lambda^{\ell-1} + \lambda^{\ell-2} + \cdots + \lambda + 1)uv^{\ell-1}.$$

Proof. If the assertion is true for $\ell > 2$, then

$$\begin{aligned} x \cdot (v^{\ell+1}) &= (g \cdot (v^\ell))(x \cdot v) + (x \cdot (v^\ell))v \\ &= (\lambda v)^\ell u + (\lambda^{\ell-1} + \lambda^{\ell-2} + \cdots + 1)uv^\ell \\ &= (\lambda^\ell + \lambda^{\ell-1} + \lambda^{\ell-2} + \cdots + 1)uv^\ell, \end{aligned}$$

as desired. ∎

So we have, $x \cdot (v^n) = (\lambda^{n-1} + \lambda^{n-2} + \cdots + 1)uv^{n-1} = 0$, and therefore

$$A^{H_n(\lambda)} = A^g = k[u, v^n].$$

3.2.2 Actions on $k[u_1, \ldots, u_r]$ for $r \geq 3$

In the preceding subsection, we determined all possible Hopf algebra actions of Taft algebras on polynomials in two variables. In this section, we show that all Hopf algebra actions on polynomials in three or more variables reduce to actions on two variables in the following sense:

Theorem 3.2.3.

(a) *In terms of the representation of H (as an algebra), there is only one possible Jordan form for x, and in this representation, $x^2 = 0$ (and therefore, $x^n = 0$).*

(b) *Given this matrix representation, we may write the matrix representing g as:*

$$\begin{pmatrix} \mathcal{G} & 0 \\ 0 & I \end{pmatrix},$$

where I and 0 represent the identity and zero matrices of the appropriate sizes, and $\mathcal{G}$ is the 2×2 matrix that represents g in the preceding subsection, §3.2.1. That is, $\mathcal{G}$ represents g in defining a Hopf algebra action of $H_n(\lambda)$ on polynomials in two variables.

(c) $A^{H_n(\lambda)} = k[u_1, u_2^n, u_3, \ldots, u_r] \cong \left(k[u_1, u_2]^{H_n(\lambda)}\right)[u_3, \ldots, u_r]$. *That is, the fixed algebra for $A = k[u_1, \ldots, u_r]$ is just the algebra formed by appending $r - 2$ independent variables to the fixed algebra calculated for $k[u, v]$ in the preceding subsection.*

Proof of this theorem will be the goal of this subsection. As before, x cannot be represented by the zero matrix, but does have zero as its only eigenvalue. To begin, we restrict the possible representations with the following lemma.

Lemma 3.2.4. *Suppose we define a Hopf algebra action for $H_n(\lambda)$ on A. The Jordan form of the matrix representing x can have blocks no larger than 2×2.*

Proof. Without loss of generality we may assume that the non-zero Jordan blocks occur in the uppermost leftmost corner, in order of decreasing size. Suppose that x contains a block with dimension greater than 2×2. We write,

$$x \mapsto \left(\begin{array}{cccc:c} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ 0 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \\ \hdashline & & & & * \end{array}\right).$$

Let $g = (g_{ij})$ for $i, j = 1, \dots, r$. We have the products:

$$xg \mapsto \left(\begin{array}{ccc:c} g_{21} & g_{22} & g_{23} & \\ g_{31} & g_{32} & g_{33} & * \\ 0 & 0 & 0 & \\ \hdashline & * & & * \end{array}\right) \quad \text{and} \quad gx \mapsto \left(\begin{array}{ccc:c} 0 & g_{11} & g_{12} & \\ 0 & g_{21} & g_{22} & * \\ 0 & g_{31} & g_{32} & \\ \hdashline & * & & * \end{array}\right).$$

Then from the relation, $xg = \lambda gx$, we obtain that $0 = g_{21} = g_{31} = g_{32}$, $g_{33} = \lambda g_{22} = \lambda^2 g_{11}$, and $g_{23} = \lambda g_{12}$. Thus for some $\alpha, \beta, \gamma \in \mathbb{C}$, g is represented by

$$\left(\begin{array}{ccc:c} \alpha & \beta & \gamma & \\ 0 & \lambda\alpha & \lambda\beta & * \\ 0 & 0 & \lambda^2\alpha & \\ \hdashline & * & & * \end{array}\right).$$

Now consider the action of x on the product, u_1u_2.

$$\begin{aligned} x \cdot (u_1u_2) &= (g \cdot u_1)(x \cdot u_2) + (x \cdot u_1)u_2 \\ &= \alpha u_1^2, \end{aligned}$$

and

$$\begin{aligned} x \cdot (u_2u_1) &= (g \cdot u_2)(x \cdot u_1) + (x \cdot u_2)u_1 \\ &= u_1^2. \end{aligned}$$

Thus, as was the case for actions on $k[u,v]$, we have that $\alpha = 1$. Now, we let x act on the product, u_2u_3.

$$\begin{aligned} x \cdot (u_2u_3) &= (g \cdot u_2)(x \cdot u_3) + (x \cdot u_2)u_3 \\ &= (\beta u_1 + \lambda u_2)u_2 + u_1u_3 \\ &= \lambda u_2^2 + \beta u_1u_2 + u_1u_3, \end{aligned}$$

but

$$\begin{aligned} x \cdot (u_3u_2) &= (g \cdot u_3)(x \cdot u_2) + (x \cdot u_3)u_2 \\ &= (\gamma u_1 + \lambda\beta u_2 + \lambda^2 u_3)u_1 + u_2^2, \end{aligned}$$

by which we see that no matter what values we choose for β and γ, we must have that $\lambda = 1$, a contradiction. ■

Proof of Theorem 3.2.3(a). We need to show that x is represented by

$$x \mapsto \left(\begin{array}{cc:c} 0 & 1 & \\ 0 & 0 & \\ \hdashline & & 0 \end{array}\right).$$

Conversely, suppose that x has the form

$$\left(\begin{array}{cc:cc:c} 0 & 1 & & & \\ 0 & 0 & & & \\ \hdashline & & 0 & 1 & \\ & & 0 & 0 & \\ \hdashline & & & & * \end{array}\right).$$

As before, writing $g = (g_{ij})$ where $i, j = 1, \ldots, r$, and using the relation $xg = \lambda gx$, we obtain the equations $g_{22} = \lambda g_{11}$, $g_{24} = \lambda g_{13}$, $g_{42} = \lambda g_{31}$, and $g_{44} = \lambda g_{33}$. Now, consider the action of x on the product u_1u_2.

$$\begin{aligned} x \cdot (u_1u_2) &= (g \cdot u_1)(x \cdot u_2) + (x \cdot u_1)u_2 \\ &= (\sum_{\ell=1}^{r} g_{\ell 1}u_\ell)u_1, \end{aligned}$$

and

$$
\begin{aligned}
x \cdot (u_2 u_1) &= (g \cdot u_2)(x \cdot u_1) + (x \cdot u_2) u_1 \\
&= u_1^2.
\end{aligned}
$$

Hence, $g_{11} = 1$ and $g_{\ell 1} = 0$ for all $\ell \neq 1$. Similarly, applying x to the product $u_3 u_4$ yields that $g_{33} = 1$ and $g_{\ell 3} = 0$ for all $\ell \neq 3$. Then, applying x to $u_2 u_4$, we see that

$$
\begin{aligned}
x \cdot (u_2 u_4) &= (g \cdot u_2)(x \cdot u_4) + (x \cdot u_2) u_4 \\
&= (g_{12} u_1 + \lambda u_2 + \cdots + g_{r2} u_r) u_3 + u_1 u_4,
\end{aligned}
$$

but

$$
\begin{aligned}
x \cdot (u_4 u_2) &= (g \cdot u_4)(x \cdot u_2) + (x \cdot u_4) u_2 \\
&= (g_{14} u_1 + g_{34} u_3 + \lambda u_4 + \cdots + g_{r4} u_r) u_1 + u_3 u_2.
\end{aligned}
$$

Comparing coefficients in either of the terms containing $u_2 u_3$ or $u_1 u_4$, implies that $\lambda = 1$, again a contradiction. ■

Proof of Theorem 3.2.3(b). Now that we know the Jordan form of x, we can explicitly determine g. As before, the relation $xg = \lambda g x$ yields the following:

$$
g \mapsto \left(\begin{array}{cc:c} \alpha & \beta & * \\ 0 & \lambda\alpha & \\ \hdashline \multicolumn{2}{c:}{*} & * \end{array} \right).
$$

Then, if we assume that $H_n(\lambda)$ acts on A and consider $x \cdot (u_1 u_2)$ and $x \cdot (u_2 u_1)$, we observe that we must have $0 = g_{i1}$ for all $i \geq 2$, and $g_{11} = 1$ (thus $g_{22} = \lambda$). Now, we consider the general product, $u_2 u_j$, when $j \geq 3$. In this case,

$$
\begin{aligned}
x \cdot (u_2 u_j) &= (g \cdot u_2)(x \cdot u_j) + (x \cdot u_2) u_j \\
&= u_1 u_j,
\end{aligned}
$$

and

$$\begin{aligned} x \cdot (u_j u_2) &= (g \cdot u_j)(x \cdot u_2) + (x \cdot u_j) u_2 \\ &= (\sum_{i=1}^{r} g_{ij} u_i) u_1. \end{aligned}$$

Therefore, $g_{ij} = \delta_{ij}$ for all $j \geq 3$, where δ_{ij} is the Kronecker delta. Hence, g has the form:

$$\begin{pmatrix} 1 & a_1 & & & \\ & \lambda & & & \\ & a_3 & 1 & & \\ & \vdots & & \ddots & \\ & a_r & & & 1 \end{pmatrix}.$$

This will give rise to an action on A since $x \cdot (u_i u_j) = 0$ for all choices of $i, j \neq 2$ and we have computed explicitly the action on products when either $i = 2$ or $j = 2$. Notice that the vectors,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{a_1}{\lambda-1} \\ 1 \\ \frac{a_3}{\lambda-1} \\ \vdots \\ \frac{a_r}{\lambda-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are linearly independent eigenvectors of g corresponding to the eigenvalues $1, \lambda, 1, \ldots, 1$ respectively. One can check,

$$\begin{bmatrix} 1 & a_1 & & & \\ & \lambda & & & \\ & a_3 & 1 & & \\ & \vdots & & \ddots & \\ & a_r & & & 1 \end{bmatrix} \begin{bmatrix} \frac{a_1}{\lambda-1} \\ 1 \\ \frac{a_3}{\lambda-1} \\ \vdots \\ \frac{a_r}{\lambda-1} \end{bmatrix} = \begin{bmatrix} a_1 + \frac{a_1}{\lambda-1} \\ \lambda \\ a_3 + \frac{a_3}{\lambda-1} \\ \vdots \\ a_r + \frac{a_r}{\lambda-1} \end{bmatrix} = \lambda \begin{bmatrix} \frac{a_1}{\lambda-1} \\ 1 \\ \frac{a_3}{\lambda-1} \\ \vdots \\ \frac{a_r}{\lambda-1} \end{bmatrix},$$

and the others are easy. Thus if we consider the algebra $A = k[u_1, \frac{a_1}{\lambda-1}u_1 + u_2 + \frac{a_3}{\lambda-1}u_3 + \cdots + \frac{a_r}{\lambda-1}u_m, u_3, \ldots, u_m] \cong k[u_1, \ldots, u_r]$, we see that the representation for g can be written as

$$g \mapsto \begin{pmatrix} 1 & & & & \\ & \lambda & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

the desired diagonalized matrix. ■

Proof of Theorem 3.2.3(c). As was the case in the previous subsection, checking the actions of x on the new generators for A gives the same representation,

$$x = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right).$$

First, we calculate A^g. It is easy to see that if $i \neq 2$, then $g \cdot u_i = u_i = \varepsilon(g)u_i$. However, we can show by induction that $g \cdot (u_2^j) = (g \cdot u_2)^j = (\lambda u_2)^j$ and hence $A^g = k[u_1, u_2^n, u_3, \ldots, u_r]$.

In order to compute $A^{H_n(\lambda)}$ we simply must find which generators of A^g are fixed by x. That is, for which $a \in A^g$ do we have $x \cdot a = \varepsilon(x)a = 0$. Clearly, for any $i \neq 2$, we have $x \cdot u_i = 0$ so that all such $u_i \in A^{H_n(\lambda)}$. Analogously to Lemma 3.2.2 we can also show that

$$x \cdot (u_2^\ell) = (\lambda^{\ell-1} + \lambda^{\ell-2} + \cdots + \lambda + 1)u_1 u_2^{\ell-1}.$$

Thus, $x \cdot (u_2^\ell) \neq 0$ for any $\ell < n$, but $x \cdot (u_2^n) = 0$. Therefore,

$$A^{H_n(\lambda)} = A^g = k[u_1, u_2^n, u_3, \ldots, u_r].$$

Hence, $A^{H_n(\lambda)} \cong \left(k[u_1, u_2]^{H_n(\lambda)}\right)[u_3, \ldots, u_r]$. ■

Chapter 4: Reflection groups and Hopf algebras

Our major goal is to describe a generalization of group actions on rings. Throughout the rest of this thesis, our principal motivation will be a generalization of the Shephard-Todd-Chevalley Theorem, a classical result describing the invariant subalgebras of polynomial rings under group actions.

4.1 Reflection groups, integrals, and trace functions

Let V be a finite dimensional complex vector space. A transformation $T : V \to V$ of a finite subgroup of $GL(V)$ is called a **pseudoreflection** if it fixes a codimension one subspace of V (and is not the identity transformation). In practice, this means that T has all but one of its eigenvalues equal to one, and the last equal to some other $\lambda \in \mathbb{C}$. Let G be a finite group of transformations on V. If additionally, G is generated by pseudoreflections, then G is called a **complex reflection group**.

Theorem 4.1.1 (Shephard-Todd-Chevalley Theorem)**.** *Let G be a group of automorphisms acting on a polynomial algebra $A = k[x_1, \ldots, x_n]$. Then the ring of invariants A^G is isomorphic to a polynomial algebra if and only if G is a complex reflection group.* ∎

One method of generalization first replaces the polynomial ring with a non-commutative analog. A finite group of automorphisms acts on an **Artin-Schelter regular** algebra A and one considers when the fixed algebra A^G is Artin-Schelter regular. This is a generalization to non-commutative algebras since the only commutative Artin-Schelter regular algebras are polynomials. En route to a full generalization to Hopf algebra actions, Kirkman, Kuzmanovich, and Zhang investigate when A^H is **Artin-Schelter Gorenstein** (a slightly weaker condition than Artin-Schelter regular) provided that A is Artin-Schelter regular. For precise definitions of Artin-Schelter regular and Artin-Schelter Gorenstein algebras as well as their formulation in this context see [1] and [10]. Henceforth we will simply say "regular" when we mean "Artin-Schelter regular." Suppose that A is regular and H is a finite-dimensional semisimple Hopf algebra acting on A, a complete generalization of Shephard-Todd-Chevalley to Hopf algebra actions must answer:

Question 4.1.2. *What are the properties of the action of H on A such that the invariant algebra A^H is regular? By analogy to Theorem* 4.1.1*, such a Hopf algebra H will be called a* ***quantum reflection group****.*

Before continuing, we give a few definitions and results for H-module algebras which are analogous to useful ideas in the case of group actions. From this point forward, we will be mostly interested with semi-simple Hopf algebras (that is, their underlying algebra structure has no nilpotent ideals). Many of the following results should be viewed as a justification for this specialization.

Definition 4.1.3. *Let H be a Hopf algebra. A* **left integral** *in H is an element $t \in H$ such that $ht = \varepsilon(h)t$, for all $h \in H$.* **Right integrals** *are defined analogously. $\int_H^l$ denotes the space of left integrals and $\int_H^r$ denotes the space of right integrals. If $\int_H^l = \int_H^r$, then H is called* **unimodular** *and we simply write $\int_H$ for the space of integrals.*

Example 4.1.4. When $H = kG$, then $t = \sum_{g \in G} g$ is a left and right integral for H. This is not difficult to check using that $\varepsilon(g) = 1$ for all $g \in G$ and that $gh = gk$ implies $h = k$ for all $g, h, k \in G$ since the elements of G are invertible. ■

Theorem 4.1.5. *Let H be a finite dimensional Hopf algebra. Then the following are equivalent.*

(i) H is semisimple.

(ii) H is unimodular.

(iii) $\varepsilon(\int_H^l) \neq 0$.

(iv) $\varepsilon(\int_H^r) \neq 0$.

Proof. It turns out that *(i)* and *(ii)* are equivalent even in the infinite dimensional case. For a full proof, see Montgomery [18]. ■

Recall that when a finite group G acts on an algebra A we can define a trace function $tr : A \to A^G$ by $a \mapsto \sum_{g \in G} g \cdot a$. We have already commented that $t = \sum_{g \in G} g$ is a left (right integral) in $H = kG$. This suggests the following generalization of trace to Hopf algebra actions.

Lemma 4.1.6. *Let H be a finite dimensional Hopf algebra, A an H-module algebra, and $0 \neq t \in \int_H^l$. Then the map $\hat{t} : A \to A$ defined by $a \mapsto t \cdot a$ is an A^H-bimodule map with values in A^H (Note: here "M is an R-bimodule" means M is both a left and right R-module.)*

Proof. Checking that $\hat{t}$ is a bimodule map is straightforward. We verify that the values are indeed in A^H. Let $a \in A$, $h \in H$. Then

$$h \cdot (\hat{t}(a)) = h \cdot (t \cdot a) = (ht) \cdot a = (\varepsilon(h)t) \cdot a = \varepsilon(h)(t \cdot a) = \varepsilon(h)(\hat{t}(a)),$$

showing $\hat{t}(a) \in A^H$. ■

Definition 4.1.7. *A map $\hat{t} : A \to A^H$ as above is called a* **left trace function** *for H on A.*

Remark 4.1.8. Now suppose that H is semisimple. Then by Theorem 4.1.5 we may choose $t \in \int_H$ such that $\varepsilon(t) = 1$. Then for any $a \in A^H$ we have that $\hat{t}(a) = t \cdot a = \varepsilon(t)a = a$. Then, it follows that the image $\hat{t}(A) = A^H$. That is, in the case of finite dimensional semisimple Hopf algebras, the trace function is always surjective. ■

Now we make more precise some of the properties that are important about the algebras we consider. Let A be a **locally finite $\mathbb{N}$-graded algebra**, that is

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

where $A_i A_j \subseteq A_{i+j}$ and $1 \in A_0$ with $\dim(A_i) < \infty$ for all $i, j \in \mathbb{N}$. Additionally, when $A_0 = k$ the algebra is said to be **connected**. Notice that polynomial algebras and the quantum plane are both connected locally finite $\mathbb{N}$-graded algebras.

Definition 4.1.9. *The Hilbert series of A is*

$$H_A(t) = \sum_{n=0}^{\infty} \dim(A_n) t^n$$

As a formal power series, the Hilbert series measures how quickly the algebra is growing as the degree increases. Stated another way, $H_A(t)$ tells how many independent generators there are in each degree. The computation of this series can uniquely determine the algebra in question. In particular, consider the following example.

Example 4.1.10. Let $A = k[x, y]$ with $\deg(x) = \deg(y) = 1$, that is, a polynomial algebra in 2 variables. Notice that in degree zero there is one generator since $A_0 = k$, in degree one we have $A_1 = \langle x, y \rangle$, in degree two there are three generators $A_2 = \langle x^2, y^2, xy \rangle$, and so on. Thus we have

$$H_A(t) = 1 + 2t + 3t^2 + 4t^3 + \cdots = \frac{1}{(1-t)^2}.$$

In fact, A is the only commutative algebra which is 2-generated in degree one with this Hilbert series. Now suppose instead that $\deg(y) = 2$. Then it is not difficult to show that

$$H_A(t) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + \cdots = \frac{1}{(1-t)(1-t^2)}.$$

We comment that there are two terms in the denominators above, each corresponding to one of the generators x and y. The quantity $(1-t^2)$ in the second series corresponds to the fact that $\deg(y) = 2$. ■

We end this section with another generating function that will be useful in the calculation of our examples. There is another notion of trace in the realm of formal power series. Suppose that H acts on a graded algebra A, then for $h \in H$ one can define the **trace** of h as the series

$$\mathrm{Tr}(h,t) = \sum_n \mathrm{tr}(h|_{A_n})t^n,$$

where $\mathrm{tr}(h|_{A_n})$ denotes the trace of the linear transformation induced by h by restricting to the degree n component subspace A_n. When H has finite dimension and is semisimple, we know that for $\tau \in \int_H$ such that $\varepsilon(\tau) = 1$, the associated trace function $\hat{\tau} : A \to A^H$ is surjective. Using this fact, Kirkman, Kuzmanovich, and Zhang extend a well-known theorem of Molien (regarding invariant algebras under group actions) to the case of Hopf algebras.

Theorem 4.1.11 (Molien's Theorem)**.** *Let H be a finite-dimensional semisimple Hopf algebra acting on the connected locally finite $\mathbb{N}$-graded algebra A such that A is an H-module algebra. Then $H_{A^H}(t) = \mathrm{Tr}(\tau, t)$ where $\tau \in \int_H$ with $\varepsilon(\tau) = 1$.*

Proof. See [10], and for more context see [8]. ∎

4.2 Semisimple Hopf algebras of dimension 8

Larson and Radford [13] proved that a semisimple Hopf algebra H of odd dimension ≤ 19 over any field k is both commutative and cocommutative. Therefore, as we have mentioned earlier, if k is algebraically closed, H is isomorphic to $kG \cong (kG)^*$, for an abelian group G. Further, we have demonstrated (Proposition 2.4.3) that the same is true for H with dimension $= 2$, since $H \cong kC_2 \cong (kC_2)^*$, and the same conclusion holds for dimension 4. Masuoka [15] has shown that there are no nontrivial semisimple Hopf algebras of dimension $= 6$, but there is one such isomorphism class of Hopf algebras having dimension 8. The unique (and hence self-dual) semisimple

Hopf algebra of dimension 8, henceforth denoted H_8, that is neither commutative nor cocommutative, is generated as an algebra by x, y, z with relations,

$$x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad yx = xy, \quad zx = yz, \quad zy = xz.$$

The coalgebra structure and antipodes are,

$$\begin{aligned} &\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \varepsilon(x) = \varepsilon(y) = 1, \\ &\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \quad \varepsilon(z) = 1, \\ &S(x) = x, \quad S(y) = y, \quad S(z) = z. \end{aligned}$$

4.2.1 Actions on $k[u_1, \ldots, u_n]$

We wish to find examples for actions of H_8 on algebras A, and we will begin by considering the case when A is isomorphic to a polynomial ring, $k[u_1, \ldots, u_n]$, of n independent variables. In this subsection we will show that there are no such actions. We will assume that $k = \mathbb{C}$, so our first task is to find a representation of the algebra in the appropriate matrix ring, $M_n(\mathbb{C})$. Notice that since x and y commute and are automorphisms of A, we may assume that they are simultaneously diagonalized since conjugation is an automorphism of A. Furthermore, they must square to the identity, implying that each has eigenvalues consisting of only ± 1. Therefore, up to automorphism in A, we can represent x as,

$$x \mapsto \begin{pmatrix} -I_{n_1} & \\ & I_{n_2} \end{pmatrix}$$

where I_{n_i} is an identity matrix of dimension $n_i \times n_i$ and n_1 is not necessarily equal to n_2. To narrow our choices further, we have the following result.

Lemma 4.2.1. *The ideals $\langle 1 - x \rangle, \langle 1 - y \rangle,$ and $\langle x - y \rangle$ are Hopf ideals.*

Proof. We will prove the result for the latter; the others are similar. Certainly $\varepsilon(\langle x -$

$y\rangle) = 0$, and moreover,

$$\begin{aligned}\Delta(x-y) &= x \otimes x - y \otimes y \\ &= x \otimes x - y \otimes y + x \otimes y - x \otimes y \\ &= x \otimes (x-y) + (x-y) \otimes y \\ &\in H_8 \otimes \langle x-y\rangle + \langle x-y\rangle \otimes H_8,\end{aligned}$$

showing that $\langle x-y\rangle$ is a coideal. The antipode requirement holds since $S(x-y) = x-y$ so $S(\langle x-y\rangle) \subseteq \langle x-y\rangle$. ■

Remark 4.2.2. It is not difficult to check that since x (respectively y) is an automorphism, if x (resp. y) is represented by the identity matrix, then x (resp. y) acts as the identity map on monomials of all degrees in A. Thus $1-x$ (resp. $1-y$) acts as the zero map in all degrees and any action of H on A will factor through the ideal $I = \langle 1-x\rangle$ (resp. $\langle 1-y\rangle$). Similarly, if x and y act identically on degree one monomials, then since both are automorphisms, an inductive argument ensures they will act identically on monomials of all degrees. Thus $x-y$ will act as the zero map on all of A and any action of H on A will factor through the ideal $I = \langle x-y\rangle$. ■

Lemma 4.2.3. *If $\rho : H_8 \to M_n(\mathbb{C})$ is an algebra map such that $\rho(x) \neq \rho(y)$, then ρ is not a representation of H_8.*

Proof. Suppose that $\rho(x) \neq \rho(y)$. By the results above, we may represent x and y without loss of generality as

$$x \mapsto \begin{pmatrix} -I_{n_1} & & \\ & -I_{n_2} & \\ & & I_{n_3} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -I_{n_1} & & \\ & I_{n_2} & \\ & & I_{n_3} \end{pmatrix},$$

where I_{n_i} represents an identity matrix of dimension $n_i \times n_i$ and n_1, n_2, n_3 are natural numbers such that $n_1 + n_2 + n_3 = n$. We will write,

$$z \mapsto \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix},$$

where each entry is a block matrix of corresponding size to those in x and y, for example A has dimension $n_1 \times n_1$. Using the relations $zx = yz$ we observe that $C, D, E, G, H = 0$. Additionally, from $xz = zy$ we obtain $B, F = 0$. Then,

$$z^2 \mapsto \begin{pmatrix} A^2 & & \\ & 0 & \\ & & B^2 \end{pmatrix}.$$

On the other hand,

$$z^2 = \frac{1}{2}(1 + x + y - xy) \mapsto \begin{pmatrix} -I & & \\ & I & \\ & & I \end{pmatrix},$$

yielding the matrix equation $0 = I$, a contradiction. ■

Theorem 4.2.4. *The polynomial ring $k[u_1, \dots, u_n]$ cannot have the structure of a non-factorable H_8-module algebra.*

Proof. By Lemma 4.2.1 we cannot have x and y represented by the same matrix, while Lemma 4.2.3 says we must have x and y represented by the same matrix. ■

4.2.2 Actions on $k_q[u, v]$

Though we have shown that H_8 cannot be made to act on polynomial algebras, in this subsection we compute all of its actions on the algebra $k_{-1}[u, v]$. Throughout, let $A = k_{-1}[u, v]$ and $k = \mathbb{C}$. The following example is due to Kirkman, Kuzmanovich, and Zhang [10].

Example 4.2.5. One can check that the following is a representation for H_8 into 2×2 matrices:

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Further, it is easily verified that the above representation gives A the structure of an H_8-module algebra. Let $G = \langle x, y \rangle$ denote the group generated by the group-like elements and A^G be the algebra of invariants under the action of kG. Some computation yields that $A^G = k[a,b,c]$ where

$$a \leftrightarrow u^3v - uv^3, \quad b \leftrightarrow u^2v^2, \quad c \leftrightarrow u^2 + v^2, \tag{4.1}$$

and the generators satisfy the relation $4b^2 - bc^2 - a^2 = 0$. Using the generalized version of Molien's Theorem and the concept of surjective trace, it can then be demonstrated that the full invariant algebra is given by

$$A^{H_8} = k[a,c].$$

Note that this algebra is abstractly isomorphic to a polynomial ring in two variables since $ac = ca$. Thus, we should regard H_8 as a "quantum reflection group" in this case, even though the representation given above does not generate a reflection group in the classical sense. In fact, even the group $G(H_8) = \langle x, y \rangle$ is not generated by classical pseudoreflections. ■

The following example is similar, only using a different representation for the Hopf algebra. Later, we will verify that the representation below and the one used in the above example are the only representations which make A an H_8-module algebra.

Example 4.2.6. It is not difficult to check that the following is a representation of H_8 in 2×2 matrices.

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad z \mapsto \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

With the above representation, we claim that A has the structure of an H_8-module algebra. Indeed, since $\Delta(x) = x \otimes x$ and $\Delta(y) = y \otimes y$, one can check that $x \cdot (vu) = -x \cdot (uv)$, and $y \cdot (vu) = -y \cdot (uv)$. To check that the action of z preserves the relation, recall that $\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$. Then,

$$\begin{aligned} z \cdot (vu) &= \frac{1}{2}\left((z \cdot v)(z \cdot u) + (z \cdot v)(xz \cdot u) + (yz \cdot v)(z \cdot u) - (yz \cdot v)(xz \cdot u)\right) \\ &= \frac{1}{2}\left((iu)(-iv) + (iu)(-iu) + (-iv)(-iv) - (-iv)(-iu)\right) \\ &= \frac{1}{2}\left(uv + u^2 - v^2 + vu)\right) \\ &= \frac{1}{2}\left(u^2 - v^2\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} z \cdot (uv) &= \frac{1}{2}\left((z \cdot u)(z \cdot v) + (z \cdot u)(xz \cdot v) + (yz \cdot u)(z \cdot v) - (yz \cdot u)(xz \cdot v)\right) \\ &= \frac{1}{2}\left((-iv)(iu) + (-iv)(iv) + (iu)(iu) - (iu)(iv)\right) \\ &= \frac{1}{2}\left(vu + v^2 - u^2 + uv\right) \\ &= \frac{1}{2}\left(v^2 - u^2\right), \end{aligned}$$

showing that, indeed, $z \cdot (vu) = -z \cdot (uv)$. Now, we compute the algebra of invariants, A^{H_8}. We begin by noting which elements of A are fixed by the group-like elements $x, y \in H_8$. Notice that $G = \langle x, y \rangle$ is represented exactly as in Example 4.2.5. Thus $A^G = k[a, b, c]$ where a, b, c are given as in (4.1). Checking the action of z on these elements gives

$$z \cdot a = -a, \quad z \cdot b = b - \frac{1}{4}a^2, \quad z \cdot c = c.$$

Hence, we have that $c \in A^{H_8}$ and $z \cdot b \in A^G$. Furthermore, we note that $k[a, b, c]$ is a commutative ring and let $B = k[a^2, c]$. We claim that $A^{H_8} = B$. First, note that indeed $a^2 \in A^{H_8}$ since,

$$\begin{aligned} z \cdot a^2 &= \frac{1}{2}((z \cdot a)(z \cdot a) + (z \cdot a)(xz \cdot a) + (yz \cdot a)(z \cdot a) - (yz \cdot a)(xz \cdot a)) \\ &= \frac{1}{2}(a^2 + a^2 + a^2 - a^2) \\ &= a^2. \end{aligned}$$

Now suppose that $\alpha \in A^{H_8}$ and $\beta \in A$. In this case, we can show that $z \cdot (\alpha\beta) = \alpha(z \cdot \beta)$. For since $\alpha \in A^{H_8}$,

$$z \cdot (\alpha\beta) = \frac{1}{2}(\alpha(z \cdot \beta) + \alpha(xz \cdot \beta) + \alpha(z \cdot \beta) - \alpha(xz \cdot \beta)) = \alpha(z \cdot \beta).$$

A similar calculation gives that $z \cdot ab = -a(z \cdot b)$. Moreover, we have that $B = k[a^2, c] \subseteq A^{H_8} \subseteq A^G = k[a, b, c] = B[a][b]$. Since a, b, c satisfy the relation $b^2 - \frac{1}{4}c^2 b - \frac{1}{4}a^2 = 0$ we can write any element $d \in A^G$ in the form

$$d = (f_1(a^2, c) + g_1(a^2, c)a) + (f_2(a^2, c) + g_2(a^2, c)a)b,$$

for some polynomials $f_i, g_j \in B$. Now suppose that $d \in A^{H_8}$ and consider the result of $z \cdot d$. Using the results from our previous calculations, we observe that

$$\begin{aligned} z \cdot d &= z \cdot (f_1 + g_1 a + f_2 b + g_2 ab) \\ &= f_1 + g_1(z \cdot a) + f_2(z \cdot b) + g_2(z \cdot ab) \\ &= f_1 - g_1 a + f_2(z \cdot b) - g_2 a(z \cdot b) \\ &= f_1 - g_1 a + (f_2 - g_2 a)(z \cdot b) \\ &= f_1 - g_1 a + (f_2 - g_2 a)(b - \frac{1}{4}a^2). \end{aligned}$$

Now if d is to be invariant under z, collecting terms with like degree gives the system of equations

$$\begin{aligned} f_1 &= f_1 - \frac{1}{4}f_2 a^2 \\ g_1 &= -g_1 + \frac{1}{4}g_2 a^2 \\ g_2 &= -g_2, \end{aligned}$$

from which we immediately obtain g_1, g_2 and f_2 are identically zero. Hence $B = A^{H_8}$. Again, notice that as in Example 4.2.5, the invariant algebra is regular. ■

Proposition 4.2.7. *The only representations which endow A with the structure of a non-factorable H_8-module algebra are given in Examples* 4.2.5 *and* 4.2.6.

Proof. The proof is by brute force calculation. We begin by noting that since x is group-like (and hence an automorphism) it must be represented by either a diagonal or skew diagonal matrix, and moreover, x must square to the identity. Then without loss of generality, in the diagonal case we are free to assume

$$x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

while in the skew case

$$x \mapsto \begin{pmatrix} 0 & \zeta \\ \frac{1}{\zeta} & 0 \end{pmatrix}$$

for some $\zeta \in \mathbb{C}$. When x is diagonal, this is because we cannot have $x \mapsto I$ or $x \mapsto -I$. In the latter scenario, we would then have $y \mapsto I$ after some calculation. The only other possibility is

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is equivalent after the automorphism exchanging the generators in $A = k_{-1}[u, v]$. In the skew case, we may improve our situation further by the automorphism given by $u \mapsto u$, $v \mapsto \zeta v$. With the new generators in A, x is represented by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Next we compute all possible representations in these two cases.

Case I: x is represented diagonally

Setting up a system of equations using the algebra relations and computing the solutions in Maple gives the following possible representations of H_8 into 2×2 matrices:

$$(i)\ y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } z \mapsto \begin{pmatrix} (\pm)i & 0 \\ 0 & (\pm)1 \end{pmatrix},$$

$$(ii)\ y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } z \mapsto \begin{pmatrix} 0 & \alpha \\ \frac{1}{\alpha} & 0 \end{pmatrix}.$$

Notice that in (i) we have x and y represented by the same matrix. Hence by Lemma 4.2.1 even if this gives rise to an action, it will factor through $\langle x - y \rangle$. Now, suppose that (ii) gives A the structure of an H_8-module algebra. Some computation yields that $z \cdot (uv) = uv$ while we also have $z \cdot (vu) = uv$, a contradiction since $vu = -uv$. Thus, there are no actions on A when x is represented diagonally.

Case II: x is represented by a skew diagonal matrix

Assuming x is represented with 1 along the off diagonal, another system of equations in Maple yields the following solutions:

(i) $y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $z \mapsto \pm \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$ where $2\alpha^2 - 2\alpha + 1 = 0$,

(ii) $y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $z \mapsto \begin{pmatrix} -\gamma & \beta \\ -\beta & \gamma \end{pmatrix}$ such that $\beta^2 = \gamma^2 - 1$.

In (i), y is represented by the same matrix as x, so any induced action is factorable. We claim that (ii) gives rise to both actions we have computed as examples in this subsection. In order to check that (ii) endows A with the structure of an H_8-module algebra, it is sufficient to check the action of z on uv and vu (the rest of the relations follow from the module requirement and the fact that x and y act as automorphisms). After some computation, we obtain the relations

$$\begin{aligned} z \cdot (uv) &= \frac{1}{2}[u^2(-\gamma^2 + 2\gamma\beta + \beta^2) + v^2(\gamma^2 + 2\gamma\beta - \beta^2)] \\ z \cdot (vu) &= \frac{1}{2}[u^2(\gamma^2 + 2\gamma\beta - \beta^2) + v^2(-\gamma^2 + 2\gamma\beta + \beta^2)]. \end{aligned}$$

Now, imposing the requirement $z \cdot (vu) = -z \cdot (uv)$, we obtain $\gamma\beta = 0$. When $\beta = 0$ we have that $\gamma^2 = 1$ giving rise to the action in Example 4.2.5. On the other hand, when $\gamma = 0$, we have $\beta^2 = -1$ giving rise to the action in Example 4.2.6. Finally, for completeness we note that we have not considered the negatives of the representations for z in Examples 4.2.5 and 4.2.6, but it is not difficult to check that these give rise to the same H_8-module structures. ■

4.3 Semisimple Hopf algebras of dimension 12

As alluded to in the previous section, the classification of finite dimensional semisimple Hopf algebras is still a topic of ongoing investigation. In addition to results of the previous section, Masuoka [16] has proven the following theorem.

Theorem 4.3.1. *Let H be a semisimple Hopf algebra over an algebraically closed field k, with $\mathrm{ch}(k) = 0$. Let $\dim(H) = 2p$ with p an odd prime. Then H, as Hopf algebra, is isomorphic to one of kC_{2p}, kD_{2p} or $(kD_{2p})^*$. Where D_{2n} denotes the dihedral group of order $2n$.* ■

In particular, this implies that there are no nontrivial Hopf algebras of dimensions 10 or 14. Recall we have already considered odd orders ≤ 19, as well as orders $2, 4, 6$ and 8 in the previous sections.

For Hopf algebras of dimension less than 20, this leaves only dimension $12, 16,$ and 18. Suppose that k is algebraically closed of characteristic $\mathrm{ch}(k) \neq 2, 3$ (in particular, what follows is true in characteristic zero). In this case Fukuda [6] has shown that there exist exactly two distinct isomorphism classes of semisimple Hopf algebras of dimension 12 which are neither commutative nor cocommutative. Per Fukuda's notation, we will denote these Hopf algebras as $\mathcal{A}_+$ and $\mathcal{A}_-$. As algebras, $\mathcal{A}_+ \cong \mathcal{A}_-$, and both are generated by appending a central idempotent element to the group algebra kS_3. That is, the algebras $\mathcal{A}_\pm$ are generated by σ, τ, υ with relations

$$\sigma^3 = 1, \quad \tau^2 = \upsilon^2 = 1, \quad \sigma\tau = \tau\sigma^2, \quad a\upsilon = \upsilon a \ (a \in \mathcal{A}_\pm).$$

The coalgebra structures and antipodes of $\mathcal{A}_+$ (respectively $\mathcal{A}_-$) are given by:

$$\Delta(\sigma) = \sigma\upsilon \otimes \sigma + \sigma(1-\upsilon) \otimes \sigma^2, \quad \varepsilon(\sigma) = 1,$$
$$\Delta(\tau) = \tau \otimes \tau \quad (\text{resp. } \tau\upsilon \otimes \tau + \tau(1-\upsilon) \otimes \tau(2\upsilon - 1)), \quad \varepsilon(\tau) = 1,$$
$$\Delta(\upsilon) = \upsilon \otimes \upsilon + (1-\upsilon) \otimes (1-\upsilon), \quad \varepsilon(\upsilon) = 1,$$
$$S(\sigma) = \sigma(1-\upsilon) + \sigma^2\upsilon,$$
$$S(\tau) = \tau \quad (\text{resp. } \tau(2\upsilon - 1)), \quad S(\upsilon) = \upsilon.$$

Given this structure, it is not difficult to prove that

Lemma 4.3.2. *The element $\mu \in \mathcal{A}_\pm$ given by*

$$\mu = \frac{1}{6}\upsilon(1 + \sigma + \sigma^2 + \tau + \tau\sigma + \tau\sigma^2)$$

is a left integral. Since $\mathcal{A}_\pm$ is semisimple and $\varepsilon(\mu) = 1$, this implies the associated trace function $\hat{\mu}$ is surjective.

Proof. Consider left multiplication by the generators:

$$\begin{aligned}
\sigma\mu &= \sigma(\frac{1}{6}\upsilon(1 + \sigma + \sigma^2 + \tau + \tau\sigma + \tau\sigma^2)) \\
&= \frac{1}{6}\upsilon(\sigma + \sigma^2 + 1 + \sigma\tau + \sigma\tau\sigma + \sigma\tau\sigma^2) \\
&= \frac{1}{6}\upsilon(\sigma + \sigma^2 + 1 + \tau\sigma^2 + \tau + \tau\sigma) \\
&= \mu = \varepsilon(\sigma)\mu.
\end{aligned}$$

The calculations for υ and τ are similar. ∎

We also note that Fukuda has shown that both $\mathcal{A}_\pm$ are self dual. Furthermore, if H is a semisimple Hopf algebra of dimension 12, it must be the case that $|G(H)| = 4$, and in particular, $G(\mathcal{A}_+) \cong C_2 \times C_2$ while $G(\mathcal{A}_-) \cong C_4$. This fact is not difficult to check. In fact, for both $\mathcal{A}_{\pm}$ it is easy to see that $2\upsilon - 1$ is group-like:

$$\begin{aligned}
\Delta(2\upsilon-1) &= 2\Delta(\upsilon) - \Delta(1)\\
&= 2\upsilon \otimes \upsilon + 2(1-\upsilon)\otimes(1-\upsilon) - 1\otimes 1\\
&= 2\upsilon \otimes \upsilon + 2(1\otimes 1) - 2(1\otimes \upsilon) - 2(\upsilon\otimes 1) + 2(\upsilon\otimes\upsilon) - 1\otimes 1\\
&= 4(\upsilon\otimes\upsilon) - 2(1\otimes\upsilon) - 2(\upsilon\otimes 1) + 1\otimes 1\\
&= 2\upsilon\otimes(2\upsilon-1) - 1\otimes(2\upsilon-1)\\
&= (2\upsilon-1)\otimes(2\upsilon-1).
\end{aligned}$$

Then $G(\mathcal{A}_+) = \{1, 2\upsilon-1, \tau, \tau(2\upsilon-1)\}$. Moreover, $(2\upsilon-1)^2 = 1$ and $\tau^2 = 1$ so $G(\mathcal{A}_+) \cong C_2 \times C_2$. In $\mathcal{A}_-$ one can check that the element $g = i\tau + (1-i)\tau\upsilon$ is group-like and that $g^2 = 2\upsilon - 1$. Thus, $G(\mathcal{A}_-) = \langle i\tau + (1+i)\tau\upsilon\rangle \cong C_4$.

Shortly, we will want to compute representations for $\mathcal{A}_{\pm}$ which allow us to endow algebras with the structure of an $\mathcal{A}_{\pm}$-module algebra.

Lemma 4.3.3. *Let $A = \bigoplus_n A_n$ be an $\mathcal{A}_{\pm}$-module algebra. If σ is represented by the identity matrix; that is, σ acts as the identity map on A_0, then $\sigma = id_A$. The same result holds for υ.*

Proof. Suppose that $a \in A$ is a monomial with $\deg(a) = d \geq 2$. Then we may write $a = a_1 a_2$ where each a_i is a monomial of degree strictly less than d. We induct on d to obtain,

$$\begin{aligned}
\sigma \cdot a_1 a_2 &= (\sigma\upsilon \cdot a_1)(\sigma \cdot a_2) + (\sigma(1-\upsilon)\cdot a_1)(\sigma^2 \cdot a_2)\\
&= (\upsilon\sigma \cdot a_1)a_2 + ((1-\upsilon)\sigma \cdot a_1)a_2 && (\upsilon \text{ is central})\\
&= (\upsilon \cdot a_1)a_2 + ((1-\upsilon)\cdot a_1)a_2\\
&= ((\upsilon + 1 - \upsilon)\cdot a_1)a_2 && (A \text{ is a Hopf module})\\
&= a_1 a_2.
\end{aligned}$$

To show the analogous result for υ, we simply note that when υ is represented by the identity matrix, $(1-\upsilon) \mapsto 0$. Thus

$$\upsilon \cdot a_1 a_2 = (\upsilon \cdot a_1)(\upsilon \cdot a_2) + ((1-\upsilon) \cdot a_1)((1-\upsilon) \cdot a_2) = a_1 a_2 + 0.$$

Again, the result follows by induction on the degree of a. ■

Lemma 4.3.4. *$I = \langle 1-\sigma \rangle$ and $J = \langle 1-\upsilon \rangle$ are Hopf ideals in both algebras, $\mathcal{A}_\pm$.*

Proof. First we check that I is a coideal. I satisfies the counit requirement, since $\varepsilon(1-\sigma) = 0$. Moreover,

$$\begin{aligned}
\Delta(1-\sigma) &= 1 \otimes 1 - \sigma\upsilon \otimes \sigma - \sigma(1-\upsilon) \otimes \sigma^2 \\
&= 1 \otimes 1 - \sigma\upsilon \otimes \sigma - \sigma \otimes \sigma^2 + \sigma\upsilon \otimes \sigma^2 \\
&= 1 \otimes 1 - \sigma\upsilon \otimes (\sigma - \sigma^2) - \sigma \otimes \sigma^2 \\
&= 1 \otimes 1 - 1 \otimes \sigma + 1 \otimes \sigma - \sigma \otimes \sigma^2 - \sigma\upsilon \otimes (1-\sigma)\sigma \\
&= 1 \otimes (1-\sigma) + 1 \otimes \sigma - 1 \otimes \sigma^2 + 1 \otimes \sigma^2 - \sigma \otimes \sigma^2 - \sigma\upsilon \otimes (1-\sigma)\sigma \\
&= 1 \otimes (1-\sigma) + 1 \otimes (1-\sigma)\sigma + (1-\sigma) \otimes \sigma^2 - \sigma\upsilon \otimes (1-\sigma)\sigma,
\end{aligned}$$

where the first, second and last terms in the final line above are elements of $\mathcal{A}_\pm \otimes I$ and the third term is an element of $I \otimes \mathcal{A}_\pm$. Therefore, $\Delta(I) \subset I \otimes \mathcal{A}_\pm + \mathcal{A}_\pm \otimes I$. We now need only to show that $S(I) \subseteq I$. Consider

$$\begin{aligned}
S(1-\sigma) &= S(1) - S(\sigma) \\
&= 1 - (\sigma(1-\upsilon) + \sigma^2\upsilon) \\
&= (1-\sigma) + (1-\sigma)\sigma\upsilon \\
&= (1-\sigma)(1-\sigma\upsilon) \in I.
\end{aligned}$$

Therefore, $\langle 1-\sigma\rangle$ is a Hopf ideal. For J, we check

$$\begin{aligned}\Delta(1-\upsilon) &= 1\otimes 1-(\upsilon\otimes\upsilon+(1-\upsilon)\otimes(1-\upsilon))\\ &= -\upsilon\otimes\upsilon+1\otimes\upsilon+\upsilon\otimes 1-\upsilon\otimes\upsilon\\ &= (1-\upsilon)\otimes\upsilon+\upsilon\otimes(1-\upsilon)\end{aligned}$$

showing $\Delta(J)\subseteq J\otimes\mathcal{A}_{\pm}+\mathcal{A}_{\pm}\otimes J$. The counit and antipode conditions are clear. ■

Thus, if we define an action of $\mathcal{A}_{\pm}$ on A in which either σ or υ act like the identity matrix on degree one monomials, then the action is factorable. In $\mathcal{A}_{+}$ we can restrict even further.

Lemma 4.3.5. *$I=\langle 1-\tau\rangle$ is a Hopf ideal of $\mathcal{A}_{+}$.*

Proof. In $\mathcal{A}_{+}$, τ is group-like and the verification is straightforward. ■

4.3.1 Actions of $\mathcal{A}_{\pm}$ on $k[x_1,\ldots,x_n]$

In this subsection we prove the following result:

Theorem 4.3.6. *The polynomial algebra $A=k[x_1,\ldots,x_n]$ cannot be endowed with the structure of a non-factorable $\mathcal{A}_{\pm}$-module algebra for any n.*

Lemma 4.3.7. *Let $A=k[x_1,\ldots,x_n]$ be a polynomial ring and suppose that $\mathcal{A}_{\pm}$ acts on A. If υ is represented by the zero matrix, then σ is represented by the identity matrix.*

Proof. Begin by choosing generators $x_1,\ldots,x_n$ such that υ and τ are diagonal. This is possible since $\upsilon\tau=\tau\upsilon$ and τ has finite order, hence is diagonalizable. The conjugation achieving this diagonal form must be an automorphism of A. Consider a generator x_i. Then $\upsilon\cdot x_i=0$ and therefore,

$$\sigma\cdot x_i^2=(\sigma\upsilon\cdot x_i)(\sigma\cdot x_i)+(\sigma(1-\upsilon)\cdot x_i)(\sigma^2\cdot x_i)=(\sigma\cdot x_i)(\sigma^2\cdot x_i).$$

Similarly, $\sigma^2 \cdot x_i^2 = (\sigma^2 \cdot x_i)(\sigma \cdot x_i)$ so that $\sigma \cdot x_i^2 = \sigma^2 \cdot x_i^2$ since A is a commutative ring. Acting on the left and right sides of this equation with σ^2 gives that $\sigma \cdot x_i^2 = x_i^2$. Similarly, $\sigma^2 \cdot x_i^2 = x_i^2$. Now, write

$$\sigma \cdot x_i = \sum_j a_j x_j, \quad \text{and} \quad \sigma^2 \cdot x_i = \sum_\ell b_\ell x_\ell.$$

Then we must have

$$x_i^2 = \sigma \cdot x_i^2 = (\sigma \cdot x_i)(\sigma^2 \cdot x_i) = \left(\sum_j a_j x_j\right)\left(\sum_\ell a_\ell x_\ell\right).$$

We immediately obtain the equation $a_i b_i = 1$ by considering the coefficient on the term x_i^2 on the right. In particular, both a_i and b_i are nonzero. Next we compute the coefficient of the term $x_i x_j$ for $j \neq i$ and obtain the equation $0 = a_i b_j + a_j b_i$. Multiplying through by b_i gives the relation $b_j = -a_j b_i^2$. Then consider the coefficient on x_j^2 for $j \neq i$. We must have $0 = a_j b_j = a_j(-a_j b_i^2)$ after substitution, and we see that $a_j^2 = 0$ for all $j \neq i$. Thus $a_j = 0$ for all such j and moreover, this implies $b_j = 0$. Hence we have that $\sigma \cdot x_i = a_i x_i \neq 0$ and $\sigma^2 \cdot x_i = b_i x_i \neq 0$. This must be true for all x_i, and therefore both σ and σ^2 are represented by diagonal matrices.

Denote the matrix representing σ by S and the matrix representing τ by T. Since both are diagonal, they must commute and we have the matrix equation $ST = TS = S^2T$ where we have also made use the relation $\sigma^2\tau = \tau\sigma$. Since T is invertible, we must have that $S = I$. ■

Proof of Theorem 4.3.6. Suppose that A is an $\mathcal{A}_\pm$-module algebra. After choosing generators so that υ and τ are simultaneously diagonalized (as above) Lemmas 4.3.4 and 4.3.7 allow us to write, without loss of generality, the following block matrices representing τ and υ:

$$\tau \mapsto \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad \text{and} \quad \upsilon \mapsto \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where D_1 and D_2 are both diagonal matrices such that each entry on the diagonal is equal to ± 1. Write $\{x_1, \dots, x_m\}$ and $\{x_{m+1}, \dots, x_n\}$ for the set of generators such that $\upsilon \cdot x_\ell = x_\ell$ for $1 \leq \ell \leq m$ and $\upsilon \cdot x_\ell = 0$ for $m < \ell \leq n$. Choose any $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, n\}$ and consider

$$\sigma \cdot x_i x_j = (\sigma\upsilon \cdot x_i)(\sigma \cdot x_j) + (\sigma(1-\upsilon) \cdot x_i)(\sigma^2 \cdot x_j) = (\sigma \cdot x_i)(\sigma \cdot x_j).$$

On the other hand, taking advantage of commutativity we have

$$\sigma \cdot x_i x_j = \sigma \cdot x_j x_i = (\sigma\upsilon \cdot x_j)(\sigma \cdot x_i) + (\sigma(1-\upsilon) \cdot x_j)(\sigma^2 \cdot x_i) = (\sigma \cdot x_j)(\sigma^2 \cdot x_i).$$

Therefore $0 = (\sigma \cdot x_j)((\sigma \cdot x_i) - (\sigma^2 \cdot x_i))$. Suppose that $\sigma \cdot x_j = 0$. Then the matrix representing σ must have the zero vector as its j-th column. However, this matrix must cube to I by the relation $\sigma^3 = 1$, a contradiction. Thus we conclude that $\sigma \cdot x_i = \sigma^2 \cdot x_i$ and acting on both sides with σ^2 gives that $\sigma \cdot x_i = x_i$. This must be true for all $i \in \{1, \dots, m\}$ and we have the following representation:

$$\sigma \mapsto \begin{bmatrix} I & S_1 \\ 0 & S_2 \end{bmatrix}, \tau \mapsto \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad \text{and} \quad \upsilon \mapsto \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

where corresponding blocks have identical dimensions. Using the relation $\sigma\upsilon = \upsilon\sigma$, we see that $S_1 = 0$. Now, since $\tau \cdot x_i = \pm x_i$ for all i, the action of $\mathcal{A}_\pm$ on $k[x_1, \dots, x_n]$ induces an action on $k[x_{m+1}, \dots, x_n]$ with the following representation:

$$\sigma \mapsto S_2, \quad \tau \mapsto D_2, \quad \text{and} \quad \upsilon \mapsto 0.$$

Then, by Lemma 4.3.7 we must have that $S_2 = I$. Therefore $S = I$, giving rise to a factorable action. ■

Combining the results of this section with those in §4.2, we have proven the following theorem:

Theorem 4.3.8. *Let H be a nontrivial semisimple Hopf algebra over an algebraically closed field of characteristic zero with* $\dim H \leq 15$. *Then for any $n \geq 2$, there are no* *non-factorable actions of H on the polynomial algebra $A = k[x_1, \ldots, x_n]$ which make A an H-module algebra.* ■

4.3.2 Actions of $\mathcal{A}_+$ on $k_q[x, y]$

Let $A = k_q[x, y]$. In this subsection, we will consider non-factorable actions of $\mathcal{A}_+$, and we begin by considering representations into 2×2 matrices over $\mathbb{C}$. Let

$$\upsilon \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{so that} \quad 1 - \upsilon \mapsto \begin{bmatrix} 1-a & -b \\ -c & 1-d \end{bmatrix}.$$

Apply the action of υ to obtain

$$\upsilon \cdot yx = (2ab - b)x^2 + (qad + 2bc + q(1-a)(1-d))xy + (2cd - c)y^2$$

$$\upsilon \cdot xy = (2ab - b)x^2 + (ad + 2qbc + (1-a)(1-d))xy + (2cd - c)y^2.$$

Requiring that $\upsilon \cdot yx = q(\upsilon \cdot xy)$ and equating coefficients gives the conditions,

$$\begin{array}{rl} x^2: & q = 1 \text{ or } b(2a-1) = 0 \\ y^2: & q = 1 \text{ or } c(2d-1) = 0 \\ xy: & q^2 = 1 \text{ or } bc = 0. \end{array} \tag{4.2}$$

We have already considered the case when $q = 1$ in the previous subsection. Now suppose $q^2 \neq 1$. Then (4.2) and the algebra relation $\upsilon^2 = \upsilon$ together provide a system of equations in a, b, c and d. Computing the solutions in Maple gives only the following as possible representations for υ:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Per our previous results (Lemma 4.3.4), the first representation (in which υ is the identity matrix) will give rise to a factorable action. Also up to interchanging the generators of A, the second and third matrices above are the same. Moreover, we note that τ is an automorphism and must be represented diagonally when $q^2 \neq 1$, reducing without loss of generality to the following two possible representations,

$$\sigma \mapsto \begin{bmatrix} r & s \\ t & u \end{bmatrix}, \quad \tau \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\upsilon \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \upsilon \mapsto 0$$

Consider the first representation. From the relations $\sigma\upsilon = \upsilon\sigma$, $\sigma^3 = 1$, and $\sigma\tau = \tau\sigma^2$ we obtain a system of equations in r, s, t, u. Some computation gives the only solution as $r = u = 1, s = t = 0$. That is, $\sigma \mapsto I$. By Lemma 4.3.4, this representation factors the action of $\mathcal{A}_+$ on A.

When υ is represented by the zero matrix, there is one additional solution. The corresponding representation for σ is

$$\sigma \mapsto \begin{bmatrix} -\frac{1}{2} & -\frac{3}{4\xi} \\ \xi & -\frac{1}{2} \end{bmatrix}, \quad \sigma^2 \mapsto \begin{bmatrix} -\frac{1}{2} & \frac{3}{4\xi} \\ -\xi & -\frac{1}{2} \end{bmatrix}. \tag{4.3}$$

Notice that if σ acts on the product $yx = qxy$ we obtain

$$\sigma \cdot yx = \frac{3}{8\xi}x^2 + \left(\frac{3}{4} + \frac{1}{4}q\right) xy + \frac{\xi}{2}y^2,$$
$$\sigma \cdot xy = -\frac{3}{8\xi}x^2 + \left(\frac{3}{4}q + \frac{1}{4}\right) xy - \frac{\xi}{2}y^2,$$

where in particular we must have $\frac{3}{8\xi} = -q\frac{3}{8\xi}$ after equating coefficients of x^2. Thus we must have $q = -1$ showing that (4.3) cannot give rise to an action on $k_q[x, y]$, when $q^2 \neq 1$.

The only remaining case is $q = -1$. By Proposition 3.1.13 we must have that the automorphism τ is represented by either a diagonal or skew diagonal matrix. As before, we begin by considering representations for υ. Using that υ is idempotent and the equations (4.2), Maple gives five solutions to the resulting system of equations. They are given below:

$$\upsilon \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} \frac{1}{2} & \frac{1}{4\xi} \\ \xi & \frac{1}{2} \end{bmatrix}, \ (\xi \neq 0).$$

Again we will not consider the first representation $\upsilon \mapsto I$ and the second and third representations are equivalent up to exchanging the variables. Next, we use Maple to check for representations of the full algebra when υ is represented by either

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} \frac{1}{2} & \frac{1}{4\xi} \\ \xi & \frac{1}{2} \end{bmatrix}, \ (\xi \neq 0).$$

For υ given by either matrix above, Maple gives that the only representations for the full algebra are those in which $\sigma \mapsto I$. Recall that we do not wish to consider such representations (Lemma 4.3.4) since they will give rise to factorable actions. Therefore, we must have that that υ is represented by the zero matrix.

Lemma 4.3.9. *When υ is represented by the zero matrix, υ acts as the identity map on monomials of even degree and the zero map on monomials of odd degree.*

Proof. If $a \in A$ is a monomial of degree one, then $\upsilon \cdot a = 0$. If a has degree two, then it can be written as the product of two degree one monomials, $a = bc$. Then

$$\upsilon \cdot a = \upsilon \cdot bc = (\upsilon \cdot b)(\upsilon \cdot c) + ((1-\upsilon) \cdot b)((1-\upsilon) \cdot c) = 0 + bc = a.$$

The result follows by induction. ∎

Suppose that τ is skew diagonal. Some computation gives that only the following representation for $\mathcal{A}_+$ (up to an automorphism scaling the generators of A) gives rise to a non-factorable action:

$$\sigma \mapsto \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad \sigma^2 \mapsto \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \tau \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \upsilon \mapsto 0 \tag{4.4}$$

For brevity, we will not include the calculations for this example, but note the theoretical result is similar to what follows.

Finally, suppose instead that τ is represented by a diagonal matrix. A Maple computation yields that the following is the only representation for $\mathcal{A}_+$ into 2×2 matrices (up to automorphism in A):

$$\sigma \mapsto \begin{bmatrix} -\frac{1}{2} & -\frac{3}{4\xi} \\ \xi & -\frac{1}{2} \end{bmatrix}, \quad \sigma^2 \mapsto \begin{bmatrix} -\frac{1}{2} & \frac{3}{4\xi} \\ -\xi & -\frac{1}{2} \end{bmatrix}, \quad \tau \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \upsilon \mapsto 0 \tag{4.5}$$

where ξ is any nonzero complex number. Note that exchanging ξ with its negation simply transposes the representation for σ with that of σ^2 in (4.5). These two elements are completely symmetric in terms of their relations in the Hopf algebra, so henceforth we will assume that ξ is in the right half-plane, $\{z \in \mathbb{C} : \Im(z) \geq 0\}$. In particular, if ξ is real, then ξ is positive. It is not difficult to check that this representation gives rise to an action on A. To aid in computation later, we have the following lemma.

Lemma 4.3.10. *Let a, b be monomials in A with even degree at least 2. When $\mathcal{A}_+$ is represented as in (4.5), we have the following relation: $\sigma \cdot ab = (\sigma \cdot a)(\sigma \cdot b)$.*

Proof. Making use of the Lemma 4.3.9, we have

$$\begin{aligned}\sigma \cdot ab &= (\sigma\upsilon \cdot a)(\sigma \cdot b) + (\sigma(1-\upsilon) \cdot a)(\sigma^2 \cdot b)\\ &= (\sigma \cdot a)(\sigma \cdot b) + 0\\ &= (\sigma \cdot a)(\sigma \cdot b),\end{aligned}$$

as desired. ∎

Example 4.3.11. Let $\mathcal{A}_+$ be represented as in (4.5). The action on degree one monomials is given by:

$$\begin{aligned}&\sigma \cdot x = -\tfrac{1}{2}x + \xi y, \quad \tau \cdot x = -x, \quad \upsilon \cdot x = 0\\ &\sigma \cdot y = -\tfrac{3}{4\xi}x - \tfrac{1}{2}y, \quad \tau \cdot y = y, \quad \upsilon \cdot y = 0\end{aligned}$$

Notice that both x^2 and y^2 are fixed under the actions of τ and υ. Hence, we have

$$(A^\upsilon)^\tau = k[x^2, y^2] = T,$$

a commutative polynomial ring. The actions of x^2 and y^2 under σ are

$$\begin{aligned}\sigma \cdot x^2 &= 0 + (\sigma(1-\upsilon) \cdot x)(\sigma^2 x)\\ &= \frac{1}{4}x^2 + \xi xy - \xi^2 y^2.\end{aligned}$$

and

$$\begin{aligned}\sigma \cdot y^2 &= 0 + (\sigma(1-\upsilon) \cdot y)(\sigma^2 y)\\ &= -\frac{9}{16\xi^2}x^2 + \frac{3}{4\xi}xy + \frac{1}{4}y^2.\end{aligned}$$

We now have enough information to begin calculation of the fixed algebra under this action. Suppose a generic element of degree two, $\alpha x^2 + \beta y^2 \in T$, for some scalars α, β, is also invariant under the action of σ. Then we obtain a system of equations from the relation $\alpha x^2+\beta y^2 = \sigma\cdot(\alpha x^2+\beta y^2)$. Maple gives the solution, $\beta = -4\xi^2\alpha/3$. Thus we write $a = 3x^2 - 4\xi^2 y^2$, as a generator of the invariant algebra. A similar calculation on degree four elements emits no new solutions; in particular, the only degree four invariant element is the square of a.

We now seek generators of degree six. If such an element is to be an independent generator, that is, not a power of a, we can without loss of generality assume that any y^6 term can be subtracted off with an appropriate multiple of a^3. Thus we write $b = \alpha x^6 + \beta x^4y^2 + \gamma x^2y^4$ for some scalars α, β, γ. Again, we solve the system obtained from $b = \sigma \cdot b$. Using Lemma 4.3.10 it is not difficult to compute the action of σ on b, and using Maple to solve the resulting equations, obtain the generator $b = x^6 + 8\xi^2x^4y^2 + 16\xi^4x^2y^4$. Then we have,

$$B = k[a, b] \subseteq A^{\mathcal{A}_+}$$

At this point, it is natural to inquire whether or not the two generators a and b are algebraically independent. Let $R = k[a]$ denote the algebra of all polynomials over k in a. Suppose that b satisfies a polynomial, $f \in R[t]$. We will show that f must be the zero polynomial by inducting on the degree of f. We cannot have f be a nonzero linear polynomial in b, by our construction of the element b, found above.

When $\deg f \geq 2$, we begin by showing that f can have no constant term in R. Suppose $0 = f(b) = r_0 + r_1b + \cdots + r_nb^n$ with $r_i \in R$ and $r_n \neq 0$. Write $r_0 = \alpha_0 + \alpha_1 a + \cdots + \alpha_m a^m$ for scalars $\alpha_j \in k$. Without loss of generality, we assume that $\alpha_m \neq 0$. Since $a = 3x^2 - 4\xi^2y^2$, after expanding r_0 in terms of x and y, there is a term (obtained from $\alpha_m a^m$) containing y^{2m}, but no other powers of x or y. This term cannot be canceled by any other term of $f(b)$ since no power of

$b = x^6 + 8\xi^2 x^4 y^2 + 16\xi^4 x^2 y^4$ contains terms consisting solely of powers in y. Hence, $\alpha_m = 0$. Similarly, we must have $\alpha_j = 0$, for all other j. Thus, $r_0 = 0$. Then we have

$$0 = f(b) = r_1 b + \cdots + r_n b^n.$$

We may divide both sides through by b to obtain $0 = r_1 + r_2 b + \cdots + r_n b^{n-1}$, a polynomial over R, satisfied by b, with degree strictly less than f. By our inductive hypothesis, $r_1 = r_2 = \cdots = r_n = 0$, and f must also be the zero polynomial. Hence, there are no relations between the elements a and b.

We conjecture that the algebra $B = k[a, b]$ obtained above *is equal to* the invariant subalgebra $A^{\mathcal{A}_+}$. Now $\deg(a) = 2$ and $\deg(b) = 6$, and standard calculation in formal power series yields that the Hilbert series of B is given by:

$$H_B(t) = 1 + t^2 + t^4 + 2t^6 + 2t^8 + 2t^{10} + 3t^{12} + \cdots = \frac{1}{(1-t^2)(1-t^6)}. \tag{4.6}$$

We will use Molien's Theorem (4.1.11) to calculate the Hilbert series of the fixed algebra. If they are the same, then $B = A^{\mathcal{A}_+}$ since B is the unique commutative algebra having the Hilbert series in (4.6). Recall (Lemma 4.3.2) that the element $\mu = \frac{1}{6}\upsilon(1 + \sigma + \sigma^2 + \tau + \tau\sigma + \tau\sigma^2)$ is a left integral for $\mathcal{A}_+$ with $\varepsilon(\mu) = 1$. Thus we have

$$H_{A^{\mathcal{A}_+}}(t) = \mathrm{Tr}(\mu, t) = \sum_{n=0}^{\infty} \mathrm{tr}(\mu|_{A_n}) t^n.$$

Now μ will act as the zero map on A_n for n odd since υ acts the zero map on monomials of odd degree. Thus, $\mathrm{tr}(\mu|_{A_m}) = 0$ for m odd and we need only compute $\mathrm{tr}(\mu|_{A_n})$ for even integers n. Thus, we may restrict to the subalgebra $T = k[x^2, y^2] = (A^{\upsilon})^{\tau} \subseteq A$ for the purposes of calculation. We can simplify our calculation further. As an example, consider

$$\begin{aligned}\mu \cdot x^2 &= \frac{1}{6}(1+\sigma+\sigma^2+\tau+\sigma\tau+\sigma^2\tau)\upsilon \cdot x^2\\ &= \frac{1}{6}(1+\sigma+\sigma^2+\tau+\sigma\tau+\sigma^2\tau) \cdot x^2\\ &= \frac{1}{6}((1+\sigma+\sigma^2)\cdot x^2 + (1+\sigma+\sigma^2)\tau \cdot x^2\\ &= \frac{1}{6}(2(1+\sigma+\sigma^2)\cdot x^2)\\ &= \frac{1}{3}(1+\sigma+\sigma^2)\cdot x^2.\end{aligned}$$

Similarly, we may inductively argue that for any $d \in T$ that $\mu \cdot d = \frac{1}{3}(1+\sigma+\sigma^2)\cdot d$. Therefore, for any positive number n,

$$\begin{aligned}\operatorname{tr}(\mu|_{T_{2n}}) &= \frac{1}{3}\left(\operatorname{tr}(1|_{T_{2n}}) + \operatorname{tr}(\sigma|_{T_{2n}}) + \operatorname{tr}(\sigma^2|_{T_{2n}})\right)\\ &= \frac{1}{3}\left((n+1) + \operatorname{tr}(\sigma|_{T_{2n}}) + \operatorname{tr}(\sigma^2|_{T_{2n}})\right).\end{aligned}$$

However, we can do even better.

Lemma 4.3.12. *For all $n \geq 0$, we have* $\operatorname{tr}(\sigma|_{T_{2n}}) = \operatorname{tr}(\sigma^2|_{T_{2n}})$.

Proof. The proof is combinatorial and utilizes the anti-symmetry in the actions of σ and σ^2 on x^2 and y^2. ■

After this simplification we have that $\operatorname{tr}(\mu|_{T_{2n}}) = \frac{1}{3}(1+n+2\operatorname{tr}(\sigma|_{T_{2n}}))$. If we are to reproduce the series in (4.6), we must show that as n takes on the values $0, 1, 2, \ldots$ that $\operatorname{tr}(\sigma|_{T_{2n}})$ follows the repeating sequence

$$1, \frac{1}{2}, 0, 1, \frac{1}{2}, 0, \ldots$$

so that for example, we want to show $\operatorname{tr}(\mu|_{T_0}) = \frac{1}{3}(1+0+2(1)) = 1$, $\operatorname{tr}(\mu|_{T_2}) = \frac{1}{3}(1+1+2(\frac{1}{2})) = 1$, and $\operatorname{tr}(\mu|_{T_4}) = \frac{1}{3}(1+2+2(0)) = 1$.

Consider T_0. It has a single generator 1_k and it easy to see that $\mathrm{tr}(\sigma|_{T_0}) = 1$. In T_2 there are two generators x^2 and y^2. Recall the action of σ on these elements:

$$\sigma \cdot x^2 = \frac{1}{4}x^2 + \xi xy - \xi^2 y^2$$
$$\sigma \cdot y^2 = -\frac{9}{16\xi^2}x^2 + \frac{3}{4\xi}xy + \frac{1}{4}y^2.$$

Thus, $\mathrm{tr}(\sigma|_{T_2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. In T_4 we have

$$\sigma \cdot x^4 = (\sigma \cdot x^2)^2 = (\frac{1}{4}x^2 + \xi xy - \xi^2 y^2)^2$$
$$\sigma \cdot x^2y^2 = (\sigma \cdot x^2)(\sigma \cdot y^2) = (\frac{1}{4}x^2 + \xi xy - \xi^2 y^2)(-\frac{9}{16\xi^2}x^2 + \frac{3}{4\xi}xy + \frac{1}{4}y^2)$$
$$\sigma \cdot y^4 = (\sigma \cdot y^2)^2 = (-\frac{9}{16\xi^2}x^2 + \frac{3}{4\xi}xy + \frac{1}{4}y^2)^2,$$

and some careful multiplication yields $\mathrm{tr}(\sigma|_{T_4}) = (\frac{1}{4})^2 + ((\frac{1}{4})^2 - \frac{3}{4} + \frac{9}{16}) + (\frac{1}{4})^2 = 0$, as desired. Some Maple worksheets are appended which verify the calculation through dimension 14. ■

Remark 4.3.13. In the example above, it is not difficult to show that the invariant algebra under the action of the group-like elements $G(\mathcal{A}_+) = \langle \tau, 2\upsilon - 1\rangle$ is $A^G = (A^\upsilon)^\tau = k[x^2, y^2] = T$. Now, if $B = k[a, b]$ is indeed the invariant subalgebra $A^{\mathcal{A}_+}$ we make the following observation. The invariant algebra is regular (it is a polynomial ring) making $\mathcal{A}_+$ a "quantum reflection group" even though σ is clearly not a reflection of T, a polynomial ring. In fact $\sigma \cdot x^2, \sigma \cdot y^2 \notin T$. Neither is σ a reflection of the original algebra $A_{-1}[x, y]$. ■

Appendix A: Maple Worksheets

The following calculations were done in Maple. One can check that these confirm the desired sequence of values for $\mathrm{tr}(\sigma|_{T_{2n}})$. That is, these calculations verify that the sequence

$$1, \frac{1}{2}, 0, 1, \frac{1}{2}, 0, \ldots$$

is correct up through dimension 14, that is $n = 7$. We have already shown these values are correct for $n = 0, 1, 2$ so these calculations begin with $n = 3$, that is, degree 6.

Denote x^2, y^2, and xy by new variables X, Y, and Z. Note that X and Y commute with everything, while Z^2= -XY. We will begin by defining the variables 'sx' and 'sy', the respective actions of sigma on X and Y.

```
> sx:=1/4*X+a*Z-a^2*Y;sy:=-9/(16*a^2)*X+3/(4*a)*Z+1/4*Y;
```

$$sx := \frac{X}{4} + a\,Z - a^2\,Y$$

$$sy := -\frac{9\,X}{16\,a^2} + \frac{3\,Z}{4\,a} + \frac{Y}{4}$$

We begin in degree 6. Consider the actions of sigma on x^6, x^4y^2, x^2y^4, y^6. By symmetry, we need only compute the action on x^4y^2. This is because the coefficient of the x^4y^2 term in s(x^4y^2) will be the same as the x^2y^4 term in s(x^2y^4). The calculations of the x^(2n) term in s(x^2n) are always easy, and are given by (1/4)^n. Similarly, those for y^(2n).

```
> f24:=collect(collect(expand(sx^2*sy^1),X),Y),Z);
```

$$f24 := \frac{3\,a\,Z^3}{4} + \left(-\frac{5\,a^2\,Y}{4} - \frac{3\,X}{16}\right) Z^2 + \left(\frac{7\,X\,a\,Y}{8} + \frac{a^3\,Y^2}{4} - \frac{15\,X^2}{64\,a}\right) Z + \frac{a^4\,Y^3}{4} - \frac{11\,X\,a^2\,Y^2}{16} - \frac{9\,X^3}{256\,a^2} + \frac{19\,X^2\,Y}{64}$$

Now we collect terms which give the coefficient of x^4y^2=X^2Y. Recall, when we use a term above containing Z^2, we must insert a negative sign.

```
> trace6:=2*(1/4)^3+2*(3/16+19/64);
```

$$trace6 := 1$$

Next we check degree 8. We need Maple to tell us the actions on x^6y^2 and x^4y^4. The rest of the information can be obtained by symmetry.

```
> f62:=collect(collect(collect(expand(sx^3*sy^1),X),Y),Z);
```

$$f62 := \frac{3\,a^2\,Z^4}{4} - 2\,a^3\,Z^3\,Y + \left(\frac{3}{2}\,a^4\,Y^2 + \frac{3}{4}\,X\,a^2\,Y - \frac{9}{32}\,X^2\right) Z^2 + \left(-\frac{3\,a^3\,X\,Y^2}{2} + \frac{3\,X^2\,a\,Y}{4} - \frac{3\,X^3}{32\,a}\right) Z - \frac{a^6\,Y^4}{4} + \frac{3\,X\,a^4\,Y^3}{4} - \frac{15\,X^2\,a^2\,Y^2}{32} + \frac{7\,X^3\,Y}{64} - \frac{9\,X^4}{1024\,a^2}$$

```
> f44:=collect(collect(collect(expand(sx^2*sy^2),X),Y),Z);
```

$$f44 := \frac{9\,Z^4}{16} + \left(-\frac{9\,X}{16\,a} - \frac{3\,a\,Y}{4}\right) Z^3 + \left(-\frac{a^2\,Y^2}{8} + \frac{21\,X\,Y}{16} - \frac{9\,X^2}{128\,a^2}\right) Z^2 + \left(-\frac{7\,X\,a\,Y^2}{16} - \frac{21\,X^2\,Y}{64\,a} + \frac{a^3\,Y^3}{4} + \frac{27\,X^3}{256\,a^3}\right) Z + \frac{a^4\,Y^4}{16} - \frac{5\,X\,a^2\,Y^3}{16} - \frac{45\,X^3\,Y}{256\,a^2} + \frac{59\,X^2\,Y^2}{128} + \frac{81\,X^4}{4096\,a^4}$$

```
> trace8:=2*(1/4)^4+2*(7/64-(-9/32))+(59/128-(21/16)+(9/16));
```

$$trace8 := \frac{1}{2}$$

Degree 10:

```
> f82:=collect(collect(collect(expand(sx^4*sy^1),X),Y),Z);
```

$$f82 := \frac{3\,a^3\,Z^5}{4} + \left(-\frac{11}{4}\,a^4\,Y + \frac{3}{16}\,a^2\,X\right) Z^4 + \left(\frac{7}{2}\,a^5\,Y^2 + \frac{1}{4}\,a^3\,X\,Y - \frac{9}{32}\,X^2\,a\right) Z^3 + \left(-\frac{3}{2}\,a^6\,Y^3 - \frac{15}{8}\,X\,a^4\,Y^2 - \frac{21}{128}\,X^3 + \frac{39}{32}\,X^2\,a^2\,Y\right) Z^2 + \left(\frac{9\,a^5\,X\,Y^3}{4} - \frac{51\,a^3\,X^2\,Y^2}{32} - \frac{33\,X^4}{1024\,a} + \frac{25\,X^3\,a\,Y}{64} - \frac{a^7\,Y^4}{4}\right) Z + \frac{a^8\,Y^5}{4} - \frac{13\,a^6\,Y^4\,X}{16} - \frac{9\,X^5}{4096\,a^2} + \frac{21\,X^2\,a^4\,Y^3}{32} - \frac{29\,X^3\,a^2\,Y^2}{128} + \frac{37\,X^4\,Y}{1024}$$

```
> f64:=collect(collect(collect(expand(sx^3*sy^2),X),Y),Z);
```

$$f64 := \frac{9\,a\,Z^5}{16} + \left(-\frac{21\,a^2\,Y}{16} - \frac{27\,X}{64}\right) Z^4 + \left(\frac{5\,a^3\,Y^2}{8} + \frac{27\,X\,a\,Y}{16} - \frac{27\,X^2}{128\,a}\right) Z^3 + \left(\frac{3\,a^4\,Y^3}{8} - \frac{57\,X\,a^2\,Y^2}{32} + \frac{45\,X^3}{512\,a^2} + \frac{9\,X^2\,Y}{128}\right) Z^2 + \left(\frac{3\,a^3\,X\,Y^3}{16} + \frac{87\,X^2\,a\,Y^2}{128} + \frac{189\,X^4}{4096\,a^3} - \frac{93\,X^3\,Y}{256\,a} - \frac{3\,a^5\,Y^4}{16}\right) Z - \frac{a^6\,Y^5}{16} + \frac{21\,X\,a^4\,Y^4}{64} + \frac{81\,X^5}{16384\,a^4} - \frac{69\,X^2\,a^2\,Y^3}{128} + \frac{149\,X^3\,Y^2}{512} - \frac{261\,X^4\,Y}{4096\,a^2}$$

```
> trace10:=2*(1/4)^5+2*(37/1024-(-21/128))+2*(149/512-(9/128)+(-27/64));
```

$$trace10 := 0$$

Degree 12:

```
> f102:=collect(collect(collect(expand(sx^5*sy^1),X),Y),Z);
```

$$f102 := \frac{3\,a^4\,Z^6}{4} + \left(-\frac{7}{2}\,a^5\,Y + \frac{3}{8}\,a^3\,X\right) Z^5 + \left(-\frac{5}{8}\,a^4\,X\,Y + \frac{25}{4}\,a^6\,Y^2 - \frac{15}{64}\,a^2\,X^2\right) Z^4 + \left(\frac{25}{16}\,a^3\,X^2\,Y - 5\,a^7\,Y^3 - \frac{5}{4}\,a^5\,X\,Y^2 - \frac{15}{64}\,X^3\,a\right) Z^3 + \left(\frac{5}{4}\,a^8\,Y^4 + \frac{55}{64}\,X^3\,a^2\,Y - \frac{105}{32}\,X^2\,a^4\,Y^2 + \frac{15}{4}\,X\,a^6\,Y^3 - \frac{75}{1024}\,X^4\right) Z^2$$

$$+\left(-\frac{21 X^5}{2048 a}+\frac{45 a^5 X^2 Y^3}{16}-\frac{25 a^7 X Y^4}{8}-\frac{65 a^3 X^3 Y^2}{64}+\frac{85 X^4 a Y}{512}+\frac{a^9 Y^5}{2}\right) Z-\frac{a^{10} Y^6}{4}+\frac{7 a^8 Y^5 X}{8}+\frac{23 X^5 Y}{2048}-\frac{55 X^2 a^6 Y^4}{64}-\frac{9 X^6}{16384 a^2}+\frac{25 X^3 a^4 Y^3}{64}-\frac{95 X^4 a^2 Y^2}{1024}$$

```
> f84:=collect(collect(collect(expand(sx^4*sy^2),X),Y),Z);
```

$$f84 := \frac{9 a^2 Z^6}{16}+\left(-\frac{15}{8} a^3 Y-\frac{9}{32} X a\right) Z^5+\left(\frac{57}{32} X a^2 Y-\frac{81}{256} X^2+\frac{31}{16} a^4 Y^2\right) Z^4+\left(\frac{45 X^2 a Y}{64}-\frac{a^5 Y^3}{4}-\frac{53 a^3 X Y^2}{16}+\frac{9 X^3}{256 a}\right) Z^3+\left(-\frac{9 a^6 Y^4}{16}-\frac{111 X^3 Y}{256}+\frac{21 X^2 a^2 Y^2}{128}+\frac{33 X a^4 Y^3}{16}+\frac{279 X^4}{4096 a^2}\right) Z^2+\left(\frac{135 X^5}{8192 a^3}-\frac{75 a^3 X^2 Y^3}{64}+\frac{3 a^5 X Y^4}{32}+\frac{211 X^3 a Y^2}{256}-\frac{411 X^4 Y}{2048 a}+\frac{a^7 Y^5}{8}\right) Z+\frac{a^8 Y^6}{16}-\frac{11 a^6 Y^5 X}{32}-\frac{171 X^5 Y}{8192 a^2}+\frac{159 X^2 a^4 Y^4}{256}+\frac{81 X^6}{65536 a^4}-\frac{109 X^3 a^2 Y^3}{256}+\frac{559 X^4 Y^2}{4096}$$

```
> f66:=collect(collect(collect(expand(sx^3*sy^3),X),Y),Z);
```

$$f66 := \frac{27 Z^6}{64}+\left(-\frac{27 a Y}{32}-\frac{81 X}{128 a}\right) Z^5+\left(\frac{243 X Y}{128}+\frac{81 X^2}{1024 a^2}+\frac{9 a^2 Y^2}{64}\right) Z^4+\left(-\frac{243 X^2 Y}{256 a}+\frac{7 a^3 Y^3}{16}-\frac{81 X a Y^2}{64}+\frac{189 X^3}{1024 a^3}\right) Z^3+\left(-\frac{3 a^4 Y^4}{64}-\frac{297 X^3 Y}{1024 a^2}+\frac{783 X^2 Y^2}{512}-\frac{33 X a^2 Y^3}{64}-\frac{243 X^4}{16384 a^4}\right) Z^2+\left(-\frac{729 X^5}{32768 a^5}-\frac{87 X^2 a Y^3}{256}+\frac{51 a^3 X Y^4}{128}-\frac{261 X^3 Y^2}{1024 a}+\frac{1377 X^4 Y}{8192 a^3}-\frac{3 a^5 Y^5}{32}\right) Z-\frac{a^6 Y^6}{64}+\frac{15 X a^4 Y^5}{128}+\frac{1215 X^5 Y}{32768 a^4}-\frac{327 X^2 a^2 Y^4}{1024}-\frac{729 X^6}{262144 a^6}+\frac{385 X^3 Y^3}{1024}-\frac{2943 X^4 Y^2}{16384 a^2}$$

```
> trace12:=2*(1/4)^6+2*(23/2048-(-75/1024))+2*(559/4096-(-111/256)+(-81/256))+(385/1024-(783/512
)+(243/128)-(27/64));
```

$$trace12 := 1$$

Degree 14:

```
> f122:=collect(collect(collect(expand(sx^6*sy^1),X),Y),Z);
```

$$f122 := \frac{3 a^5 Z^7}{4}+\left(\frac{9}{16} a^4 X-\frac{17}{4} a^6 Y\right) Z^6+\left(-\frac{15}{8} a^5 X Y-\frac{9}{64} a^3 X^2+\frac{39}{4} a^7 Y^2\right) Z^5+\left(-\frac{45}{4} a^8 Y^3+\frac{15}{16} a^6 X Y^2+\frac{105}{64} a^4 X^2 Y-\frac{75}{256} a^2 X^3\right) Z^4+\left(\frac{25}{4} a^9 Y^4-\frac{165}{32} a^5 X^2 Y^2-\frac{135}{1024} X^4 a+\frac{15}{4} a^7 X Y^3+\frac{95}{64} a^3 X^3 Y\right) Z^3+\left(-\frac{3}{4} a^{10} Y^5-\frac{117}{4096} X^5-\frac{345}{128} X^3 a^4 Y^2+\frac{465}{1024} X^4 a^2 Y+\frac{225}{32} X^2 a^6 Y^3-\frac{105}{16} X a^8 Y^4\right) Z^2+\left(\frac{129 X^5 a Y}{2048}+\frac{33 a^9 X Y^5}{8}-\frac{51 X^6}{16384 a}-\frac{285 a^7 X^2 Y^4}{64}-\frac{525 a^3 X^4 Y^2}{1024}+\frac{135 a^5 X^3 Y^3}{64}-\frac{3 a^{11} Y^6}{4}\right) Z+\frac{a^{12} Y^7}{4}-\frac{15 a^{10} Y^6 X}{16}-\frac{141 X^5 a^2 Y^2}{4096}+\frac{69 X^2 a^8 Y^5}{64}+\frac{55 X^6 Y}{16384}-\frac{155 X^3 a^6 Y^4}{256}-\frac{9 X^7}{65536 a^2}+\frac{195 X^4 a^4 Y^3}{1024}$$

```
> f104:=collect(collect(collect(expand(sx^5*sy^2),X),Y),Z);
```

$$f104 := \frac{9 a^3 Z^7}{16}+\left(-\frac{9}{64} a^2 X-\frac{39}{16} a^4 Y\right) Z^6+\left(\frac{61}{16} a^5 Y^2+\frac{51}{32} a^3 X Y-\frac{99}{256} X^2 a\right) Z^5+\left(-\frac{35}{16} a^6 Y^3-\frac{295}{64} X a^4 Y^2+\frac{375}{256} X^2 a^2 Y-\frac{45}{1024} X^3\right) Z^4+\left(-\frac{5 a^7 Y^4}{16}+\frac{315 X^4}{4096 a}-\frac{175 a^3 X^2 Y^2}{128}-\frac{75 X^3 a Y}{256}+\frac{85 a^5 X Y^3}{16}\right) Z^3+\left(\frac{11 a^8 Y^5}{16}+\frac{549 X^5}{16384 a^2}+\frac{665 X^3 a^2 Y^2}{512}-\frac{1545 X^4 Y}{4096}-\frac{105 X^2 a^4 Y^3}{128}-\frac{135 a^6 Y^4 X}{64}\right) Z^2+\left(-\frac{717 X^5 Y}{8192 a}-\frac{13 a^7 X Y^5}{32}+\frac{351 X^6}{65536 a^3}+\frac{465 a^5 X^2 Y^4}{256}+\frac{2225 X^4 a Y^2}{4096}-\frac{395 a^3 X^3 Y^3}{256}-\frac{a^9 Y^6}{16}\right) Z-\frac{a^{10} Y^7}{16}+\frac{23 a^8 Y^6 X}{64}+\frac{901 X^5 Y^2}{16384}-\frac{181 a^6 Y^5 X^2}{256}-\frac{423 X^6 Y}{65536 a^2}+\frac{595 X^3 a^4 Y^4}{1024}+\frac{81 X^7}{262144 a^4}-\frac{995 X^4 a^2 Y^3}{4096}$$

```
> f86:=collect(collect(collect(expand(sx^4*sy^3),X),Y),Z);
```

$$f86 := \frac{27\,a\,Z^7}{64} + \left(-\frac{135\,X}{256} - \frac{81\,a^2\,Y}{64}\right) Z^6 + \left(\frac{63\,a^3\,Y^2}{64} + \frac{297\,X\,a\,Y}{128} - \frac{81\,X^2}{1024\,a}\right) Z^5 + \left(\frac{19\,a^4\,Y^3}{64} - \frac{801\,X\,a^2\,Y^2}{256} - \frac{567\,X^2\,Y}{1024} + \frac{837\,X^3}{4096\,a^2}\right) Z^4$$
$$+ \left(-\frac{31\,a^5\,Y^4}{64} + \frac{513\,X^4}{16384\,a^3} + \frac{1107\,X^2\,a\,Y^2}{512} - \frac{729\,X^3\,Y}{1024\,a} + \frac{55\,a^3\,X\,Y^3}{64}\right) Z^3$$
$$+ \left(-\frac{3\,a^6\,Y^5}{64} - \frac{1701\,X^5}{65536\,a^4} + \frac{855\,X^3\,Y^2}{2048} + \frac{1809\,X^4\,Y}{16384\,a^2} - \frac{1023\,X^2\,a^2\,Y^3}{512} + \frac{231\,X\,a^4\,Y^4}{256}\right) Z^2$$
$$+ \left(\frac{3321\,X^5\,Y}{32768\,a^3} - \frac{39\,a^5\,X\,Y^5}{128} - \frac{2187\,X^6}{262144\,a^5} + \frac{123\,a^3\,X^2\,Y^4}{1024} - \frac{6741\,X^4\,Y^2}{16384\,a} + \frac{559\,X^3\,a\,Y^3}{1024} + \frac{5\,a^7\,Y^6}{64}\right) Z + \frac{a^8\,Y^7}{64} - \frac{31\,a^6\,Y^6\,X}{256} - \frac{5373\,X^5\,Y^2}{65536\,a^2}$$
$$+ \frac{357\,X^2\,a^4\,Y^5}{1024} + \frac{3159\,X^6\,Y}{262144\,a^4} - \frac{1867\,X^3\,a^2\,Y^4}{4096} - \frac{729\,X^7}{1048576\,a^6} + \frac{4483\,X^4\,Y^3}{16384}$$

```
> trace14:=2*(1/4)^7+2*(55/16384-(-117/4096))+2*(901/16384-(-1545/4096)+(-45/1024))+2*(4483/1638
  4-(855/2048)+(-567/1024)-(-135/256));
```

$$trace14 := \frac{1}{2}$$

At this point, to continue, we will have to start calculating at least four terms at a time. A more general proof that the pattern continues is required.

```
>
```

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Vita

Justin Allman was born May 17, 1982 to Dr. Richard and Connie Allman of Morgantown, WV. He graduated from Vestavia Hills High School, Birmingham, AL in May 2001 and completed a Bachelor of Science in May 2005 at the Georgia Institute of Technology, with a major in Physics and minor in Mathematics. In December of that year, he was married to Sara Katherine Robb Allman of Memphis, TN. He received a Master of Arts in Education from Wake Forest University along with secondary teaching certification in August 2006, after which he taught high school algebra and trigonometry at West Forsyth High School, in Clemmons, NC, until returning to Wake Forest in August 2007. He is a proud member of the mathematics faculty at the North Carolina Governor's School - West, housed each summer at Salem College in Winston-Salem. In May 2009, he will be awarded the degree of Master of Arts in Mathematics by Wake Forest University, and begin work towards a Doctor of Philosophy in Mathematics in August 2009 at the University of North Carolina, Chapel Hill.