A Bound for Linear Recurrence Relations with Unbounded Order

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Abstract

This paper considers linear recurrence relations with unbounded order where the coefficients are restricted to intervals of negative real numbers. The optimal inequalities, under the given constraints, exhibit a second order structure.

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1 Introduction

This paper studies general linear recurrences of the form

$$b_n = \sum_{k=1}^{n-1} \beta_{n,k} b_k, \text{ (for } n \geq 2), \tag{1.1}$$

where, for some fixed $A > B \geq 0$ and $A \geq 1$,

$$\beta_{n,k} \in [-A, -B], \tag{1.2}$$

for $1 \leq k \leq n-1$ and $n \geq 2$. Without loss of generality we will assume that $b_1 = 1$.

We are interested, here, in the structure of the bounding sequence $\{U_i\}$ defined by

$$U_n = U_n(A, B) \stackrel{def}{=} \max\{|b_n| : \{b_i\} \text{ and } \{\beta_{i,j}\} \text{ satisfy (1.1) and (1.2)}\}, \tag{1.3}$$

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for $n \geq 1$.

We will prove the following theorem.

Theorem 1 *With $\{U_j\}$ defined as in (1.3),*

$$U_n = \begin{cases} A, & \text{if } n = 2 \\ \max\{A^2 - B, A - B^2\}, & \text{if } n = 3 \\ A(A^2 - 2B + 1), & \text{if } n = 4 \\ A^4 - 3A^2B + B^2 - B, & \text{if } n = 5 \\ AU_{n-1} + (1 - B)U_{n-2}, & \text{if } n \geq 6 \end{cases}.$$

In [3], an explicit form for U_n was obtained for the complimentary case of intervals which contain zero, i.e. when (1.2) is replaced by $\beta_{n,k} \in [-A, B]$ for $A > B \geq 0$. The reader is referred to [3] and the references therein for discussion of applications to applicable bounds for reciprocal of power series and inverses of triangular matrices.

We remark that our results and those in [3] leave open the following question regarding subintervals of the negative unit interval.

Open Question. What is the value of $\{U_n\}$ when in place of (1.2), we consider (for some fixed $1 > C > D > 0$)

$$\beta_{n,k} \in [-C, -D], \tag{1.4}$$

Linear recurrences as in (1.1) arise in investigation of power series. For related results involving restricted coefficients within that realm see [1], [2], [4], [5], [6], [7] and [8].

2 Preliminaries for the proof of Theorem 1

In this section, we will provide a collection of definitions and preliminary results. Suppose $\{b_i\}$ and $\{\beta_{i,j}\}$ satisfy (1.1) and (1.2) with $b_1 = 1$.

First, for $n \geq 1$, let

$$P_n = \sum_{\substack{1 \leq k \leq n \\ b_k \geq 0}} b_k \text{ and } N_n = \sum_{\substack{1 \leq k \leq n \\ b_k < 0}} b_k. \tag{2.1}$$

Note that, $\{P_i\}$ is non-decreasing and $\{N_i\}$ is non-increasing.

The following elementary bound on b_n follows from (1.1) and (1.2).

Lemma 2 *For $n \geq 2$,*

$$-AP_{n-1} - BN_{n-1} \leq b_n \leq -BP_{n-1} - AN_{n-1} \tag{2.2}$$

and

$$-AP_{n-1} - BN_{n-1} < -BP_{n-1} - AN_{n-1}. \tag{2.3}$$

PROOF. *By (1.1) and (1.2),*

$$\begin{aligned} b_n &= \sum_{k=1}^{n-1} \beta_{n,k} b_k \\ &\geq -A \sum_{\substack{1\leq k\leq n-1 \\ b_k \geq 0}} b_k - B \sum_{\substack{1\leq k\leq n-1 \\ b_k < 0}} b_k \\ &= -AP_{n-1} - BN_{n-1}. \end{aligned}$$

The upper bound in (2.2) follows by a similar argument and (2.3) follows from

$$-BP_{n-1} - AN_{n-1} - (-AP_{n-1} - BN_{n-1}) = (A - B)(P_{n-1} - N_{n-1}) > 0.$$

Theorem 3 *We have $U_2 = A$ and $U_3 = \max\{A^2 - B, A - B^2\}$.*

PROOF. *By (1.1), we have $b_1 = 1$ and*

$$b_2 = \beta_{2,1} b_1 = \beta_{2,1} \in [-A, -B].$$

Thus, we have

$$U_2 = A$$

and

$$\begin{cases} P_2 = b_1 = 1 \\ N_2 = b_2 \in [-A, -B] \end{cases}. \tag{2.4}$$

By Lemma 2, $-AP_2 - BN_2 \leq b_3 \leq -BP_2 - AN_2$ and

$$\begin{aligned} -AP_2 - BN_2 &\geq -A(1) - B(-B) = B^2 - A \\ -BP_2 - AN_2 &\leq -B(1) - A(-A) = A^2 - B > 0. \end{aligned}$$

Hence $B^2 - A \le b_3 \le A^2 - B$.
If $B^2 - A \ge 0$, *i.e.* $A - B^2 \le 0$, *then,*

$$U_3 = A^2 - B = \max\{A^2 - B, A - B^2\}.$$

If $B^2 - A < 0$, *then*

$$U_3 = \max\{|A^2 - B|, |B^2 - A|\} = \max\{A^2 - B, A - B^2\}.$$

Note 1 *The value of* $A - B^2$ *is greater than that of* $A - B^2$ *only in the small region*

$$\{(A, B) : 0 \le B \le 1 \text{ and } 1 \le A \le \frac{1}{2}(1 + \sqrt{1 + 4B - 4B^2}\ )\}. \tag{2.5}$$

This region is shown in Figure 1.

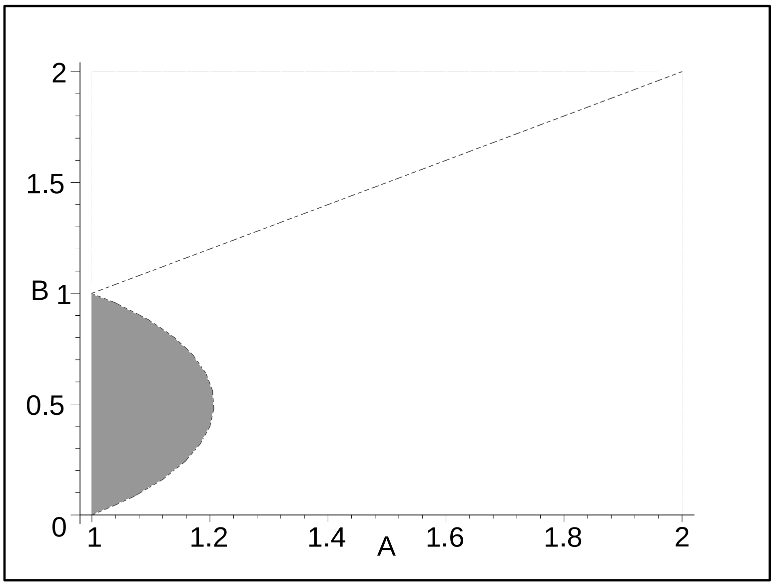

Fig. 1. The region in which $A - B^2$ is greater than $A^2 - B$

Lemma 4 *If* $B \le 1$, $n \ge 2$ *and*

$$-AP_{n-1} - BN_{n-1} \le 0 \le -BP_{n-1} - AN_{n-1}, \tag{2.6}$$

then

$$-AP_n - BN_n \le 0 \le -BP_n - AN_n. \tag{2.7}$$

PROOF. *The result is trivial for* $B = 0$.

Suppose $b_n \ge 0$. *Then* $N_n = N_{n-1}$ *and by Lemma 2, we have* $0 \le b_n \le -BP_{n-1} - AN_{n-1}$ *and*

$$P_{n-1} \le P_n \le P_{n-1} - BP_{n-1} - AN_{n-1} = (1 - B)P_{n-1} - AN_{n-1}.$$

Hence,

$$AP_n + BN_n \geq AP_{n-1} + BN_{n-1} \geq 0$$

and, since $0 \leq B \leq 1$*,*

$$\begin{aligned} BP_n + AN_n &\leq B[(1-B)P_{n-1} - AN_{n-1}] + AN_{n-1} \\ &= (1-B)(BP_{n-1} + AN_{n-1}) \\ &\leq 0. \end{aligned}$$

The case for $b_n < 0$ *follows by a similar argument.*

Corollary 5 *If* $B \leq 1$*, then*

$$-AP_n - BN_n \leq 0 \leq -BP_n - AN_n \text{ for } n \geq 2. \tag{2.8}$$

PROOF. *First by (1.1) and (1.2),* $b_1 = 1$ *and* $b_2 = \beta_{2,1} b_1 = \beta_{2,1} \in [-A, -B]$*. Thus, we have* $P_2 = b_1 = 1$ *and* $N_2 = b_2 \in [-A, -B]$*, and hence,*

$$-AP_2 - BN_2 \leq A(B-1) \leq 0 \leq B(A-1) \leq -BP_2 - AN_2.$$

Equation (2.8) follows by induction via Lemma 4.

Definition 6 *Let* $\{\tilde{b}_i\}_{i=1}^n$ *be an instance of* $\{b_i\}_{i=1}^n$ *for which* $|b_n|$ *attains the bound* U_n*. We denote the* $\{\beta_{i,j}\}$*,* $\{P_i\}$ *and* $\{N_i\}$ *corresponding to this instance by* $\{\tilde{\beta}_{i,j}\}$*,* $\{\tilde{P}_i\}$ *and* $\{\tilde{N}_i\}$*, respectively.*

Note 1 *The sequence* $\{\tilde{b}_i\}_{i=1}^n$ *is defined such that* $|\tilde{b}_n| = U_n$*, it is not required that* $|\tilde{b}_i| = U_i$ *for* $1 \leq i \leq n-1$*.*

Note 2 *Whenever* $\tilde{b}_i$ *is used in this paper, we always assume that it is coming from* $\{\tilde{b}_i\}_{i=1}^n$ *with* $|\tilde{b}_n| = U_n$*. The value of* n *should be clear from the context.*

Lemma 7 *For* $n \geq 4$*,* $\tilde{b}_n$ *and* $\tilde{b}_{n-1}$ *must have alternating signs and, specifically, we either have*

$$\tilde{b}_n = A(B-1)\tilde{P}_{n-2} + (A^2 - B)\tilde{N}_{n-2} < 0 \text{ and } \tilde{b}_{n-1} = -B\tilde{P}_{n-2} - A\tilde{N}_{n-2} \geq 0 \tag{2.9}$$

or

$$\tilde{b}_n = (A^2 - B)\tilde{P}_{n-2} + A(B-1)\tilde{N}_{n-2} > 0 \text{ and } \tilde{b}_{n-1} = -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} < 0. \tag{2.10}$$

PROOF. *From (2.2),*

$$-A\tilde{P}_{n-2} - B\tilde{N}_{n-2} \leq \tilde{b}_{n-1} \leq -B\tilde{P}_{n-2} - A\tilde{N}_{n-2}.$$

Depending on the signs of $-A\tilde{P}_{n-2} - B\tilde{N}_{n-2}$ *and* $-B\tilde{P}_{n-2} - A\tilde{N}_{n-2}$*, there are three cases to consider.*

Case 1 *(*$-A\tilde{P}_{n-2} - B\tilde{N}_{n-2} \le 0$ *and* $-B\tilde{P}_{n-2} - A\tilde{N}_{n-2} \ge 0$*).*

By (1.1),

$$\begin{aligned}
\tilde{b}_n &= \sum_{k=1}^{n-2} \tilde{\beta}_{n,k}\tilde{b}_k + \tilde{\beta}_{n,n-1}\tilde{b}_{n-1} \\
&\ge -A \sum_{\substack{1\le k\le n-2 \\ b_k \ge 0}} \tilde{b}_k - B \sum_{\substack{1\le k\le n-2 \\ b_k < 0}} \tilde{b}_k + \tilde{\beta}_{n,n-1}\tilde{b}_{n-1} \\
&\ge (-A\tilde{P}_{n-2} - B\tilde{N}_{n-2}) - A(-B\tilde{P}_{n-2} - A\tilde{N}_{n-2}) \qquad (2.11) \\
&= A(B-1)\tilde{P}_{n-2} + (A^2 - B)\tilde{N}_{n-2}.
\end{aligned}$$

Equality holds for (2.11) when $\tilde{b}_{n-1} = -B\tilde{P}_{n-2} - A\tilde{N}_{n-2} \ge 0$ *and* $\tilde{\beta}_{n,n-1} = -A$*.*

Similarly,

$$\begin{aligned}
\tilde{b}_n &= \sum_{k=1}^{n-2} \tilde{\beta}_{n,k}\tilde{b}_k + \tilde{\beta}_{n,n-1}\tilde{b}_{n-1} \\
&\le -B \sum_{\substack{1\le k\le n-2 \\ b_k \ge 0}} \tilde{b}_k - A \sum_{\substack{1\le k\le n-2 \\ b_k < 0}} \tilde{b}_k + \tilde{\beta}_{n,n-1}\tilde{b}_{n-1} \\
&\le (-B\tilde{P}_{n-2} - A\tilde{N}_{n-2}) - A(-A\tilde{P}_{n-2} - B\tilde{N}_{n-2}) \qquad (2.12) \\
&= (A^2 - B)\tilde{P}_{n-2} + A(B-1)\tilde{N}_{n-2}.
\end{aligned}$$

Equality holds for (2.12) when $\tilde{b}_{n-1} = -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} < 0$ *and* $\tilde{\beta}_{n,n-1} = -A$*.*

Note, by (2.3), the RHS in (2.11) is negative and the RHS in (2.12) is positive, and hence we either have (2.9) or (2.10).

Case 2 *(*$-A\tilde{P}_{n-2} - B\tilde{N}_{n-2} > 0$ *and* $-B\tilde{P}_{n-2} - A\tilde{N}_{n-2} > 0$*).*

By a similar argument to that in Case 1*, we either have*

$$\begin{aligned}
\tilde{b}_{n-1} &= -B\tilde{P}_{n-2} - A\tilde{N}_{n-2} > 0 \qquad (2.13) \\
\tilde{b}_n &= A(B-1)\tilde{P}_{n-2} + (A^2 - B)\tilde{N}_{n-2} \qquad (2.14) \\
&= -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} - A(-B\tilde{P}_{n-2} - A\tilde{N}_{n-2}) < 0
\end{aligned}$$

or

$$\begin{aligned}\tilde{b}_{n-1} &= -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} > 0 \qquad (2.15)\\ \tilde{b}_n &= B(A-1)\tilde{P}_{n-2} + (B^2-A)\tilde{N}_{n-2}. \qquad (2.16)\end{aligned}$$

Note that the RHS in (2.14) is more negative than that of (2.16) and thus has a larger absolute value if the latter is non-positive.

Suppose the RHS of (2.16) is positive. We have $\tilde{N}_{n-2} < -\frac{A}{B}\tilde{P}_{n-2}$ *and by Corollary 5,* $B > 1$.

Hence,

$$\begin{aligned}&|A(B-1)\tilde{P}_{n-2} + (A^2-B)\tilde{N}_{n-2}| - |B(A-1)\tilde{P}_{n-2} + (B^2-A)\tilde{N}_{n-2}|\\ &= -A(B-1)\tilde{P}_{n-2} - (A^2-B)\tilde{N}_{n-2} - B(A-1)\tilde{P}_{n-2} - (B^2-A)\tilde{N}_{n-2}\\ &= -[A(B-1)+B(A-1)]\tilde{P}_{n-2} - [A(A-1)+B(B-1)]\tilde{N}_{n-2}\\ &\geq -[A(B-1)+B(A-1)]\tilde{P}_{n-2} - [A(A-1)+B(B-1)]\left[-\frac{A}{B}\tilde{P}_{n-2}\right]\\ &= \frac{(A^2-B^2)(A-1)}{B}\tilde{P}_{n-2}\\ &\geq 0,\end{aligned}$$

and thus (2.15) and (2.16) do not hold.

Case 3 *($-A\tilde{P}_{n-2} - B\tilde{N}_{n-2} < 0$ and $-B\tilde{P}_{n-2} - A\tilde{N}_{n-2} < 0$).*

By a similar argument to that in Case 2, we have,

$$\begin{aligned}\tilde{b}_{n-1} &= -A\tilde{P}_{n-2} - B\tilde{N}_{n-2} < 0 \qquad (2.17)\\ \tilde{b}_n &= A(B-1)\tilde{N}_{n-2} + (A^2-B)\tilde{P}_{n-2} > 0. \qquad (2.18)\end{aligned}$$

Definition 8 *Let $\{\hat{b}_n\}$ be a special case of $\{b_n\}$ obtained by setting*

$$\beta_{i,j} = \hat{\beta}_{i,j} = \begin{cases} -B, \text{ if } i \equiv j \mod (2) \\ -A, \text{ if } i \not\equiv j \mod (2) \end{cases}. \qquad (2.19)$$

Lemma 9 *We have*

$$\begin{aligned}\hat{b}_1 &= 1, \qquad (2.20)\\ \hat{b}_2 &= -A, \qquad (2.21)\\ \hat{b}_3 &= A^2 - B, \qquad (2.22)\\ \hat{b}_4 &= -A(A^2-2B+1), \text{ and} \qquad (2.23)\\ \hat{b}_5 &= A^4 - 3A^2B + B^2 - B. \qquad (2.24)\end{aligned}$$

PROOF. *By direct computation.*

Note 1 *Note that* $U_1 = \hat{b}_1$ *and* $U_2 = -\hat{b}_2$. *If* (A, B) *is not in* (2.5), *we also have* $U_3 = \hat{b}_3$.

Lemma 10 *For* $n \geq 4$,

$$\hat{b}_n = -A\hat{b}_{n-1} + (1 - B)\hat{b}_{n-2}. \tag{2.25}$$

PROOF. *By Definition 8 and* (1.1),

$$\begin{aligned}\hat{b}_n &= \sum_{i=1}^{n-3} \hat{\beta}_{n,i}\hat{b}_i + \hat{\beta}_{n,n-2}\hat{b}_{n-2} + \hat{\beta}_{n,n-1}\hat{b}_{n-1} \\ &= \sum_{i=1}^{n-3} \hat{\beta}_{n-2,i}\hat{b}_i + (-B)\hat{b}_{n-2} + (-A)\hat{b}_{n-1} \\ &= \hat{b}_{n-2} + (-B)\hat{b}_{n-2} + (-A)\hat{b}_{n-1} \\ &= (1 - B)\hat{b}_{n-2} - A\hat{b}_{n-1}.\end{aligned}$$

Lemma 11 $\{\hat{b}_n\}$ *is alternating.*

PROOF. *From Lemma 9, it is not hard to see that the first three terms of* $\{\hat{b}_n\}$ *are alternating and for n=3, we have*

$$\hat{b}_n = \begin{cases} -BP_{n-1} - AN_{n-1} > 0, \text{ if } n \text{ is odd} \\ -AP_{n-1} - BN_{n-1} < 0, \text{ if } n \text{ is even} \end{cases}. \tag{2.26}$$

Assume (2.26) *is true for* $3 \leq n \leq k - 1$ *and* $\{\hat{b}_n\}_{n=1}^{k-1}$ *is alternating. Without loss of generality, suppose* k *is odd. Then* $k - 1$ *is even, and thus we have*

$$\hat{b}_{k-1} = -AP_{k-2} - BN_{k-2} < 0.$$

Definition 8 and (1.1) *give*

$$
\begin{aligned}
\hat{b}_k &= \sum_{\substack{1\le i\le k-1\\ k \text{ is odd}}} \hat{\beta}_{k,i}\hat{b}_i + \sum_{\substack{1\le i\le k-1\\ k \text{ is even}}} \hat{\beta}_{k,i}\hat{b}_i \\
&= -B \sum_{\substack{1\le i\le k-1\\ k \text{ is odd}}} \hat{b}_i - A \sum_{\substack{1\le i\le k-1\\ k \text{ is even}}} \hat{b}_i \\
&= -BP_{k-1} - AN_{k-1} \\
&= -BP_{k-2} - A(N_{k-2} + \hat{b}_{k-1}) \\
&= -BP_{k-2} - A[(N_{k-2} - AP_{k-2} - BN_{k-2}] \\
&= (-BP_{k-2} - AN_{k-2}) - A(-AP_{k-2} - BN_{k-2}),
\end{aligned}
$$

which, by the induction assumption and (2.3), is positive.

The case when k is even is similar, and the proof is finished by mathematical induction.

3 When $B \le 1$ and $n \ge 4$

In this section, we consider the case where $B \le 1$ and $n \ge 4$.

Lemma 12 *When $B \le 1$ and $n \ge 4$, $\tilde{b}_n$, $\tilde{b}_{n-1}$ and $\tilde{b}_{n-2}$ must have alternating signs, specifically we have one of the following two possiblities.*

(1) $(sgn(\tilde{b}_{n-2}, sgn(\tilde{b}_{n-1},, sgn(\tilde{b}_n)) = (+,-,+)$ *and*

$$
\begin{aligned}
\tilde{b}_{n-2} &= -B\tilde{P}_{n-3} - A\tilde{N}_{n-3} \ge 0, && (3.1)\\
\tilde{b}_{n-1} &= A(B-1)\tilde{P}_{n-3} + (A^2 - B)\tilde{N}_{n-3} && (3.2)\\
&= -A\tilde{P}_{n-3} - B\tilde{N}_{n-3} - A(-B\tilde{P}_{n-3} - A\tilde{N}_{n-3}) < 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{b}_n &= (A^2 - B)(1-B)\tilde{P}_{n-3} - A(A^2 - 2B + 1)]\tilde{N}_{n-3} && (3.3)\\
&= -A\tilde{b}_{n-1} + (1-B)\tilde{b}_{n-2} > 0; && (3.4)
\end{aligned}
$$

(2) $(sgn(\tilde{b}_{n-2}, sgn(\tilde{b}_{n-1},, sgn(\tilde{b}_n)) = (-,+,-)$ *and*

$$
\begin{aligned}
\tilde{b}_{n-2} &= -A\tilde{P}_{n-3} - B\tilde{N}_{n-3} < 0, && (3.5)\\
\tilde{b}_{n-1} &= (A^2 - B)\tilde{P}_{n-3} + A(B-1)\tilde{N}_{n-3} && (3.6)\\
&= -B\tilde{P}_{n-3} - A\tilde{N}_{n-3} - A(-A\tilde{P}_{n-3} - B\tilde{N}_{n-3}) > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{b}_n &= -A(A^2 - 2B + 1)\tilde{P}_{n-3} + (A^2 - B)(1-B)\tilde{N}_{n-3} && (3.7)\\
&= -A\tilde{b}_{n-1} + (1-B)\tilde{b}_{n-2} < 0. && (3.8)
\end{aligned}
$$

PROOF. *From Lemma 7, we know that* $\tilde{b}_n$ *and* $\tilde{b}_{n-1}$ *must have alternating signs.*

If $(sgn(\tilde{b}_{n-2}), sgn(\tilde{b}_{n-1}), sgn(\tilde{b}_n)) = (+, -, +)$*, by (2.10),*

$$\begin{aligned} |\tilde{b}_n| &= (A^2 - B)\tilde{P}_{n-2} + A(B-1)\tilde{N}_{n-2} \\ &= (A^2 - B)(\tilde{P}_{n-3} + \tilde{b}_{n-2}) + A(B-1)\tilde{N}_{n-3}. \end{aligned} \tag{3.9}$$

If $(sgn(\tilde{b}_{n-2}), sgn(\tilde{b}_{n-1}), sgn(\tilde{b}_n)) = (+, +, -)$*, by (2.9),*

$$\begin{aligned} |\tilde{b}_n| &= -[A(B-1)\tilde{P}_{n-2} + (A^2 - B)\tilde{N}_{n-2}] \\ &= -[A(B-1)(\tilde{P}_{n-3} + \tilde{b}_{n-2}) + (A^2 - B)\tilde{N}_{n-3}]. \end{aligned} \tag{3.10}$$

In both cases, a larger value for $\tilde{b}_{n-2}$ *gives a larger value for* $\tilde{b}_n$*. Thus from (2.2),* $\tilde{b}_{n-2} = -B\tilde{P}_{n-3} - A\tilde{N}_{n-3}$*. Substituting this into (3.9) and (3.10), and taking the difference gives*

$$\begin{aligned} (3.9) - (3.10) &= [(A-1)(A+B)][\tilde{P}_{n-3} + (-B\tilde{P}_{n-3} - A\tilde{N}_{n-3}) + \tilde{N}_{n-3}] \\ &= [(A-1)(A+B)][(1-B)\tilde{P}_{n-3} + (1-A)\tilde{N}_{n-3}] \\ &\geq 0, \end{aligned}$$

$(sgn(\tilde{b}_{n-2}), sgn(\tilde{b}_{n-1}), sgn(\tilde{b}_n))$ *cannot be* $(+, +, -)$*,*
and if $(sgn(\tilde{b}_{n-2}), sgn(\tilde{b}_{n-1}), sgn(\tilde{b}_n)) = (+, -, +)$*,*
then $\tilde{b}_{n-2}$ *,* $\tilde{b}_{n-1}$ *and* $\tilde{b}_n$ *are as in (3.1), (3.2) and (3.3) , respectively.*

A similar proof shows that $(sgn(\tilde{b}_{n-2}), sgn(\tilde{b}_{n-1}), sgn(\tilde{b}_n))$ *cannot be* $(-, -, +)$*,*
and if $(sgn(\tilde{b}_{n-2}), sgn(\tilde{b}_{n-1}), sgn(\tilde{b}_n)) = (-, +, -)$
then $\tilde{b}_{n-2}$*,* $\tilde{b}_{n-1}$ *and* $\tilde{b}_n$ *are as in (3.5), (3.6) and (3.7), respectively.*

Theorem 13 *When* $B \leq 1$*,*

$$U_4 = A(A^2 - 2B + 1). \tag{3.11}$$

PROOF. *By definition,* $\tilde{b}_2 \leq 0$*. Thus by Lemma 12,* $(sgn(\tilde{b}_2), sgn(\tilde{b}_3), sgn(\tilde{b}_4))$ *must be* $(-, +, -)$*. By (3.7), we have*

$$\begin{aligned} U_4 &= -[-A(A^2 - 2B + 1)\tilde{P}_1 + (A^2 - B)(1 - B)\tilde{N}_1] \\ &= -[-A(A^2 - 2B + 1)(1) + (A^2 - B)(1 - B)(0)] \\ &= A(A^2 - 2B + 1). \end{aligned}$$

Theorem 14 *When $B \leq 1$, $U_n = |\hat{b}_n|$, for $n \geq 4$.*

PROOF. *From Lemma 9 and Theorem 13, $U_4 = |\hat{b}_4|$.*

Suppose $U_i = |\hat{b}_i|$ for $4 \leq i \leq n-1$.

From Lemma 12, and more specifically (3.4) and (3.8), we know that

$$\tilde{b}_n = -A\tilde{b}_{n-1} + (1-\tilde{B})\tilde{b}_{n-2}, \tag{3.12}$$

where $\tilde{b}_{n-2}$ is of the same sign as and $\tilde{b}_{n-1}$ is of opposite sign to $\tilde{b}_n$. Hence we have

$$U_n = |\tilde{b}_n| = A|\tilde{b}_{n-1}| + (1-B)|\tilde{b}_{n-2}| \leq AU_{n-1} + (1-B)U_{n-2}. \tag{3.13}$$

Now from (3.13), and Lemmas 10 and 11, we have

$$U_n \geq |\hat{b}_n| = A|\hat{b}_{n-1}| + (1-B)|\hat{b}_{n-2}| = AU_{n-1} + (1-B)U_{n-2} \geq U_n.$$

Thus, we have $U_n = |\hat{b}_n|$. The proof is completed by induction.

This concludes the proof of Theorem 1 when $B \leq 1$ and $n \geq 4$. We now turn to the proof of Theorem 1 when $B > 1$ and $n \geq 4$.

4 When $B > 1$ and $n \geq 4$

In this section, we consider the case when $B > 1$ and $n \geq 4$.

For convenience, when $B > 1$, we modify the original problem to allow b_1 to be 1 or -1. In switching b_1 from 1 to -1, while keeping the values of all $\beta_{i,j}$ unchanged, only the signs of every term in $\{b_k\}$ change but the original moduli are preserved. Hence this modified problem yields the same U_n as the original problem and in the modified problem $U_n = \max\{b_n : b_n \geq 0\}$.

Lemma 15 *If $B > 1$ and $\tilde{b}_n \geq 0$, where $n \geq 4$, we have*

$$\tilde{b}_n = \begin{cases} (1-B)\hat{b}_m\tilde{P}_{n-m} + \hat{b}_{m+1}\tilde{N}_{n-m}, & \text{for } 0 < m < n \text{ and } m \text{ is odd} \\ \hat{b}_{m+1}\tilde{P}_{n-m} + (1-B)\hat{b}_m\tilde{N}_{n-m}, & \text{for } 0 < m < n \text{ and } m \text{ is even} \end{cases}, \tag{4.1}$$

PROOF. *From Lemma 7, we know* (4.1) *is true for m=1. Suppose* (4.1) *is true for* $1 \leq m \leq k-1$.

If k is odd, then from the induction assumption, we have

$$\tilde{b}_n = \hat{b}_k \tilde{P}_{n-k+1} + (1-B)\hat{b}_{k-1}\tilde{N}_{n-k+1}.$$

Note also that

$$\begin{cases} \tilde{P}_{n-k+1} = \tilde{P}_{n-k} + \tilde{b}_{n-k+1} & \text{and } \tilde{N}_{n-k+1} = \tilde{N}_{n-k}, \quad \text{if } b_{n-k+1} \geq 0 \\ \tilde{P}_{n-k+1} = \tilde{P}_{n-k} & \text{and } \tilde{N}_{n-k+1} = \tilde{N}_{n-k} + \tilde{b}_{n-k+1}, \text{ if } b_{n-k+1} < 0 \end{cases}.$$

From Lemma 11 and the condition $B > 1$*, we know that* $\hat{b}_k > 0$ *and* $(1-B)\hat{b}_{k-1} > 0$. *So a more positive* $\tilde{P}_{n-k+1}$ *would give a larger* $|\tilde{b}_n|$ *than a more negative* $\tilde{N}_{n-k+1}$. *Thus, we have* $\tilde{b}_{n-k+1} = -B\tilde{P}_{n-k} - A\tilde{N}_{n-k}$ *and so we have*

$$\begin{aligned} \tilde{b}_n &= \hat{b}_k(\tilde{P}_{n-k} + \tilde{b}_{n-k+1}) + (1-B)\hat{b}_{k-1}\tilde{N}_{n-k} \\ &= \hat{b}_k(\tilde{P}_{n-k} - B\tilde{P}_{n-k} - A\tilde{N}_{n-k}) + (1-B)\hat{b}_{k-1}\tilde{N}_{n-k} \\ &= (1-B)\hat{b}_k\tilde{P}_{n-k} + [-A\hat{b}_k + (1-B)\hat{b}_{k-1}]\tilde{N}_{n-k} \\ &= (1-B)\hat{b}_k\tilde{P}_{n-k} + \hat{b}_{k+1}\tilde{N}_{n-k}. \end{aligned}$$

The argument when k is even is very similar and the proof is completed by mathematical induction.

Theorem 16 *For* $n \geq 4$*, if* $B > 1$*,* $U_n = |\hat{b}_n|$.

PROOF. *In the modified problem, we only need to consider the case when* $\tilde{b}_n \geq 0$. *From* (4.1)*, we have*

$$\tilde{b}_n = \begin{cases} \hat{b}_n P_1 + (1-B)\hat{b}_{n-1}N_1, \text{ if } n \text{ is odd} \\ (1-B)\hat{b}_{n-1}P_1 + \hat{b}_n N_1, \text{ if } n \text{ is even} \end{cases}. \tag{4.2}$$

Note that (P_1, N_1) *can only be* $(1,0)$ *or* $(0,-1)$.

If n is odd, then $\hat{b}_n > 0$ *and* $(1-B)\hat{b}_{n-1} > 0$*, so we would choose* (P_1, N_1) *to be* $(1,0)$*, and hence* $\tilde{b}_n = \hat{b}_n$.

If n is even, then $\hat{b}_n < 0$ *and* $(1-B)\hat{b}_{n-1} < 0$*, so we would choose* (P_1, N_1) *to be* $(0,-1)$*, and hence* $\tilde{b}_n = \hat{b}_n$.

So combining both cases, we have $|U_n| = |\hat{b}_n|$.

5 Conclusion

Theorem 1 now follows from Theorems 3, 13, 14 and 15 and Lemmas 9 and 10.

We restate that our results leave open the case of subintervals of the negative unit interval. The results herein and in [3] give that the sequence in (1.3) eventually satisfies a second order recurrence for all other intervals. We conjecture that the same holds for the remaining cases. But it appears that novel techniques will be needed.

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